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Research School of Information Sciences and Engineering
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Australian National University

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ARITHMETIC AND TRUTH IN ŁUKASIEWICZ'S INFINITELY VALUED LOGIC

Greg Restall

Abstract Peano arithmetic formulated in Łukasiewicz's infinitely valued logic collapses into classical Peano arithmetic. However, not all additions to the language need also be classical. The way is open for the addition of a real truth predicate satisfying the T-scheme into the language. However, such an addition is not pleasing. The resulting theory is ω-inconsistent. This paper consists of the proofs and interpretations of these two results.

1. Introduction

T-sentences and unrestricted self-reference do not mix well. Given the T-scheme (that $\vdash T\ulcorner A \urcorner \leftrightarrow A$ for each sentence A, and a Gödel coding $\ulcorner - \urcorner$) and the means for self-reference (say, that given by a Gödel coding) it is simple to construct sentences like L satisfying $\vdash L \leftrightarrow \sim L$ (liar sentences) and C satisfying $\vdash C \leftrightarrow (C \to D)$ for any D we like (Curry-paradoxical sentences). Using moves like *reductio* $(A \to \sim A \vdash \sim A)$ and *contraction* $(A \to (A \to B) \vdash A \to B)$ it is simple to deduce absurd conclusions from these sentences. In addition, Löb has shown that given contraction, no provability predicate satisfying minimal conditions[1] can satisfy $\vdash Prv\ulcorner A \urcorner \to A$ [2]. These are significant limitative results in formal arithmetic.

Reductio and contraction are valid in nearly every logic on the market, so the standard reactions to these problems has been to either abandon self-reference and keep truth (by enforcing some kind of type discipline) or to keep self-reference and abandon the notion of truth. A third approach is clear. We can keep both self-reference and truth, while rejecting both contraction and *reductio*.[2]

2. Logic Without Contraction or *Reductio*

It is well known that Łukasiewicz's infinitely valued logic $\mathbf{Ł}_\infty$ is resistent to paradoxes. Neither contraction nor *reductio* are valid in $\mathbf{Ł}_\infty$, and the naïve comprehension scheme is consistent in $\mathbf{Ł}_\infty$ [6]. $\mathbf{Ł}_\infty$ seems resistant to the standard paradoxes. However, we will see that $\mathbf{Ł}_\infty$ is not a panacea for paradoxes in arithmetic. Before that, we must introduce $\mathbf{Ł}_\infty$.

The predicate logic $\mathbf{Ł}_\infty$ is a simple extension of classical predicate logic, given by expanding the set of truth values for evaluating formulae to the real interval $[0, 1]$. If we take 0 to be the designated value, then conjunction and disjunction ($\wedge$ and $\vee$) on the interval are maximum and minimum respectively. Negation ($\sim$) maps x onto $1 - x$, and the conditional ($\to$) is restricted subtraction. (A conditional with an antecedent valued x and a consequent valued y is given the value $y \dot{-} x$. This means that if the consequent is 'as true as' the antecedent the conditional is true, and if not, the conditional falls short of the truth as far as the consequent differs from the antecedent.) Finally, universal and existential quantification ($\forall$ and $\exists$) are modelled by supremum and infimum respectively. Given some first-order language $\mathcal{L}$, an $\mathbf{Ł}_\infty$ structure for that language is a domain together with an interpretation for each predicate in the language (a function from the appropriate cartesian product of the domain to the interval $[0, 1]$) and for each function symbol (a function on the domain). Truth values for arbitrary formulae are defined recursively in the usual manner. A formula is said to hold in an $\mathbf{Ł}_\infty$ structure just when it receives the value 0 in that structure. Theorems are the formulae that hold in every $\mathbf{Ł}_\infty$ structure.

It is useful to consider another connective $\circ$, called 'fusion.' $A \circ B$ is defined as $\sim(A \to \sim B)$. Fusion is the residual of the conditional, in that

$$\big((A \circ B) \to C\big) \leftrightarrow \big(A \to (B \to C)\big)$$

is a theorem of $\mathbf{Ł}_\infty$. It is simple to show that fusion is modelled by restricted addition. That is, if A has value x and B has value y, then $A \circ B$ has value $\min(1, x+y)$. Therefore, $A \circ A$ has value twice that of A (or 1, if you hit the limit), and $A \circ (A \circ A)$ thrice that of A (or 1) and so on. We write A^n for the n-fold fusion of A with itself. This has an interesting corollary:

LEMMA 1 *If a formula A does not hold in a $\mathbf{Ł}_\infty$ structure (it is not evaluated as 0) then for some n, A^n is false in that structure (it is evaluated as 1).*

Proof: If A has value x, then A^m has value $\min(1, mx)$. Given that $x \geq 0$, an integer $n \geq 1/x$ will give A^n value 1 as desired. $\triangleleft$

THEOREM 2 *Classical logic* **TV** *results from $\mathbf{Ł}_\infty$ by adding any of $A \vee \sim A$, $A \wedge (A \to B) \to B$, $A \to (A \circ A)$, and $(A \to \sim A) \to \sim A$.*

Proof: For each axiom scheme with only one sentence variable A, the scheme is only true when A is evaluated as either 1 or 0. In the scheme $A \wedge (A \to B) \to B$, if this is true for all B, then we have $A \wedge (A \to \sim A) \to \sim A$, and it is clear that this is true only when A is either 0 or 1. Any evaluation in which these schemes are true must be classical. $\triangleleft$

3. Arithmetic in $\mathbf{Ł}_\infty$

To examine behaviour of arithmetic in $\mathbf{Ł}_\infty$, we will use a set of axioms known to give Peano arithmetic in the context of classical logic, but which also do not break the 'spirit' of the $\mathbf{Ł}_\infty$ law. The axioms are couched in the first order language with one binary relation symbol '$=$', one unary constant $\underline{0}$, one unary function symbol '$'$', and two binary function symbols '$+$' and '$\times$'. (As usual, we abbreviate '$\times$' by juxtaposition.)

$$\begin{array}{rc}
\textit{Identity} & \underline{0} = \underline{0}, \quad \forall x \forall y (x = y \to y = x), \\
& \forall x \forall y \forall z \big(y = z \to (x = y \to x = z)\big), \\
& \forall x \forall y (x' = y' \to x = y), \\
\textit{Successor} & \forall x \forall y (x = y \to x' = y'), \quad \forall x (\underline{0} \neq x'), \\
\textit{Addition} & \forall x (x + \underline{0} = x), \quad \forall x \forall y \big(x + y' = (x+y)'\big), \\
\textit{Multiplication} & \forall x (x\underline{0} = \underline{0}), \quad \forall x \forall y (xy' = xy + x), \\
\textit{Induction} & A(\underline{0}), \forall x \big(A(x) \to A(x')\big) \vdash \forall x A(x).
\end{array}$$

A $\mathbf{Ł}_\infty^\#$ fact is anything true in all $\mathbf{Ł}_\infty$ structures in which all axioms hold, and in which the induction scheme is truth preserving. We write '$\mathbf{Ł}_\infty^\# \vDash A$' to indicate that A is a $\mathbf{Ł}_\infty^\#$ fact.

Note that induction is only assumed as a rule. This is because assuming the stronger axiom form $A(\underline{0}) \wedge \forall x\big(A(x) \to A(x')\big) \to \forall x A(x)$ would be out of place in a logic like Ł_∞. Cashing out the quantifiers in the axiom in terms of conjunctions we have

$$A(\underline{0}) \wedge \big(A(\underline{0}) \to A(\underline{1})\big) \wedge \big(A(\underline{1}) \to A(\underline{2})\big) \wedge \big(A(\underline{2}) \to A(\underline{3})\big) \wedge \cdots \to A(\underline{0}) \wedge A(\underline{1}) \wedge A(\underline{2}) \cdots$$

But in Ł_∞, claims like $A \wedge (A \to B) \to A \wedge B$ and $A \wedge (A \to B) \wedge (B \to C) \to A \wedge B \wedge C$, are not theorems, in general. (If they were, so would $A \to A \circ A$, and Ł_∞ would collapse into classical logic.) So, it would be out of place to posit the induction scheme in such a strong form.

The question naturally arises: how does $\text{Ł}_\infty^\#$ differ from Peano arithmetic, $\mathbf{TV}^\#$? The answer is: surprisingly little. Adhering to contraction-free scruples matters little for Peano arithmetic. We have the following result.

THEOREM 3 *For every A, $\text{Ł}_\infty^\# \vDash A$ iff $\mathbf{TV}^\# \vDash A$.*

Proving this fact involves showing that excluded middle holds in $\text{Ł}_\infty^\#$ structures. We prove this in degrees. First excluded middle for equations of the form $\underline{0} = x$, then for all equations, and finally, for all formulae.

LEMMA 4 $\text{Ł}_\infty^\# \vDash \underline{0} = x \vee \underline{0} \neq x$.

Proof: Clearly $\text{Ł}_\infty^\# \vDash \underline{0} = \underline{0} \vee \underline{0} \neq \underline{0}$, and $\text{Ł}_\infty^\# \vDash \forall x(\underline{0} = x \vee \underline{0} \neq x \to \underline{0} = x' \vee \underline{0} \neq x')$ by virtue of the fact that $\text{Ł}_\infty^\# \vDash \underline{0} \neq x'$. Induction then gives us the result. $\triangleleft$

To get the result for arbitrary equations, it is helpful to have a double induction rule.

LEMMA 5 *If $\text{Ł}_\infty^\# \vDash A(\underline{0}, y)$, $\text{Ł}_\infty^\# \vDash A(x, \underline{0})$, and $\text{Ł}_\infty^\# \vDash A(x, y) \to A(x', y')$ then $\text{Ł}_\infty^\# \vDash \forall x \forall y A(x, y)$ too.*

Proof: Firstly, $\text{Ł}_\infty^\# \vDash \forall y A(x, y) \to A(x', 0)$ as $\text{Ł}_\infty^\# \vDash A(x', 0)$. This is the base case of our first induction. For the induction step, note that we have $\text{Ł}_\infty^\# \vDash \forall y A(x, y) \to A(x', y')$, strengthening the antecedent of the double induction step we hypothesised. So, we also have

$$\text{Ł}_\infty^\# \vDash \big(\forall y A(x, y) \to A(x', y)\big) \to \big(\forall y A(x, y) \to A(x', y')\big)$$

by weakening in the antecedent. This gives $\text{Ł}_\infty^\# \vDash \forall y\big(\forall y A(x, y) \to A(x', y')\big)$ by induction on the free instances of y. Confining the quantifiers we have $\text{Ł}_\infty^\# \vDash \forall y A(x, y) \to \forall y A(x', y)$. But we have $\text{Ł}_\infty^\# \vDash \forall y A(\underline{0}, y)$ too, so an induction on x gives $\text{Ł}_\infty^\# \vDash \forall x \forall y A(x, y)$ as desired. $\triangleleft$

Then it is a simple step to excluded middle for all equations.

LEMMA 6 $\text{Ł}_\infty^\# \vDash \exists x(y = z \leftrightarrow \underline{0} = x)$, *and hence* $\text{Ł}_\infty^\# \vDash y = z \vee y \neq z$.

Proof: Note that $\text{Ł}_\infty^\# \models \exists x(\underline{0} = z \leftrightarrow \underline{0} = x)$ and $\text{Ł}_\infty^\# \models \exists x(y = \underline{0} \leftrightarrow \underline{0} = x)$. These are the two base cases of our double induction. We also have $\text{Ł}_\infty^\# \models y = z \leftrightarrow y' = z'$, so this will give us the induction step $\text{Ł}_\infty^\# \models \exists x(y = z \leftrightarrow \underline{0} = x) \rightarrow \exists x(y' = z' \leftrightarrow \underline{0} = x)$. Double induction then gives the first result.

For the second, $\text{Ł}_\infty^\# \models \exists x(y = z \leftrightarrow \underline{0} = x)$ shows us that in any structure, an arbitrary identity is equivalent to one of the form $\underline{0} = x$. This second identity is evaluated as either zero or one in every $\text{Ł}_\infty^\#$ structure, and hence, so is the original identity. This means that excluded middle holds for all identities. ◁

There is one last step to excluded middle for all formulae.

LEMMA 7 $\text{Ł}_\infty^\# \models A \vee {\sim} A$ *for each formula* A.

Proof: Every formula is recursively built up from identities by the propositional connectives and the quantifiers. The identities are interpreted as either zero or one in every structure. The connectives and quantifiers, when given classical values, only return classical values. Hence, any formula will be evaluated as either zero or one, and hence, excluded middle holds of any formula. ◁

This final lemma completes the proof of Theorem 3, given the result of Theorem 2, which tells us that adding exluded middle to Ł_∞ collapses it into classical logic. This means that in weak logics like Ł_∞, we keep the full strength of classical logic for arithmetic. We need not take classical logic as the best theory of inference, while agreeing that it is correct for reasoning about numbers. This need not be true for other predicates we may wish to add to the language. A predicate added to the language is not bound to be classical. It is required only to obey the more liberal laws of Ł_∞. This makes analysing truth and self-reference in a logic like Ł_∞ an exciting prospect. We are not forced to 'cripple' the deductive machinery of the mathematics in order to count the paradoxical inferences as invalid.

This means that even such oddities as Löb's theorem hold in $\text{Ł}_\infty^\#$, in the sense that, for any provability predicate Prv *in the language of arithmetic* if $\vdash Prv\ulcorner A \urcorner \rightarrow A$ then $\vdash A$. This is not as restrictive as it might seem, because Łukasiewicz's logic keeps the way open for there to be another provability predicate Prv^* for which Löb's result fails. For example, any truth predicate T will do for this purpose. The only restriction is that the predicate will not be expressible in the language of arithmetic: it must enrich the language.

4. The Truth about Truth in Ł_∞

To add truth into the theory, let's enrich the language of arithmetic by a unary predicate 'T' denoting truth. As a predicate of numbers, we must employ a Gödel coding to transfer reference to numbers to reference to formulas. Pick a particular coding, $\ulcorner - \urcorner$, so that for every formula A, $\ulcorner A \urcorner$ is a numeral in the language. Then we require that T pick out truth. One way to do this is to posit Tarski's scheme

$$\vdash T\ulcorner A \urcorner \leftrightarrow A$$

Adding this scheme to $\L_\infty^\#$ results in the arithmetic $\L_\infty^{\#T}$ containing truth. The rest of this paper will consist of the proof that $\L_\infty^{\#T}$ is ω-inconsistent, and the interpretation of this fact.

Before we can present the proof proper, we need a definition of ω-inconsistency appropriate for $\L_\infty^\#$. The guiding thought is that theory is ω-inconsistent if and only if its closure under the ω rule (from $A(\underline{0}), A(\underline{1}), A(\underline{2}), \ldots$ to derive $\forall x A(x)$) is inconsistent. This prompts the following definition.

Definition 1 A theory X is ω-inconsistent just when for some A and each n, $X \vDash A(\underline{n})$, but in addition, $X \vDash \sim\forall x A(x)$.

Clearly, if an arithmetic theory is ω-inconsistent, it has no standard models. As a result, if a theory is ω-inconsistent, it is not a satisfactory theory of arithmetic. The proof that $\L_\infty^{\#T}$ is ω-inconsistent is not difficult.

Firstly, we need some self-reference in order to concoct the ω-inconsistent formulae. For this, we need a special form of the diagonalisation lemma that allows for formulae with free variables to be diagonalised. Recall the diagonalisation function, which takes Gödel numbers of formulae and returns the Gödel number of its diagonalisation, given a particular free variable we've picked out for the purpose. Let the variable in our case be x, then the diagonalisation of A is $\exists x(x = \ulcorner A \urcorner \wedge A)$.

LEMMA 8 *If we have an arithmetic theory that is classical in the arithmetic fragment, and which represents the diagonal function* $diag$*, then for any formula* $B(y)$ *with at least the variable* y *free, there is a formula* R *(with at most the variables other than* y *free in* $B(y)$ *free in it) where* $\vdash R \leftrightarrow B(\ulcorner R \urcorner)$ *in that arithmetic.*

Proof: The proof is standard. Let $A(x, y)$ represent $diag$ in the arithmetic. So, for any n, k if $diag(n) = k$ then $\vdash \forall y(A(\underline{n}, y) \leftrightarrow y = \underline{k})$.

Let F be the formula $\exists y(A(x, y) \wedge B(y))$. F is a formula with at most the variables other than y free in $B(y)$ free in it. Let $n = \ulcorner F \urcorner$ and let R be the expression $\exists x(x = \underline{n} \wedge \exists y(A(x, y) \wedge B(y)))$. As $\underline{n} = \ulcorner F \urcorner$, R is the diagonalisation of F, and it has the same variables free in it as in F, except for x. So we must have $\vdash R \leftrightarrow \exists y(A(\underline{n}, y) \wedge B(y))$. Let $\underline{k} = \ulcorner R \urcorner$; then $diag(n) = k$ and we must have $\vdash \forall y(A(\underline{n}, y) \leftrightarrow y = \underline{k})$ as A represents $diag$. It follows that $\vdash R \leftrightarrow \exists y(y = \underline{k} \wedge B(y))$, which gives $\vdash R \leftrightarrow B(\underline{k})$ as desired. ◁

Now we can define our formulae to prove ω-inconsistency. Let $A_0, A_1, A_2, \ldots$ be defined as follows:[3]

$$A_0 = \sim\forall x \exists y(R(x+1, y) \wedge Ty) \qquad A_{n+1} = A_0^{n+1}$$

where R is a recursive predicate defined to represent the Gödel codes of $A_0, A_1, A_2 \ldots$. This means that Rxy is true whenever y is the Gödel code of sentence number x in the list above, and false otherwise. Such a predicate R is expressible in the arithmetic fragment of the language. Consider how all of the formluae A_i are 'manufactured' from the previous ones by a very simple procedure. Given the code of A_i, the code of A_{i+1} is found by applying a simple transformation. This can be packed into a recursive definition. We don't yet have the code $\ulcorner R \urcorner$ of R, so we'll leave this as the argument place of a function. Then we can let $h(\ulcorner R \urcorner, 0)$ be the Gödel number of A_0, as a function of whatever code we give R itself. Then, we can let $h(\ulcorner R \urcorner, n+1)$ be the Gödel number of A_{n+1}, given by a simple transformation on $h(\ulcorner R \urcorner, n)$. The resulting function is recursive. (We defined it by recursion in simple arithmetic functions dictating the way formulae are made up out of subformulae). So, this function h of two free variables can be represented by some predicate in classical arithmetic. Call this predicate H. To get the value of $\ulcorner R \urcorner$, we need it to satisfy:

$$\vdash Rxy \leftrightarrow H(\ulcorner R \urcorner, x, y)$$

But we have such an R by the diagonalisation lemma that we proved.

THEOREM 9 $\mathbf{Ł}^{\#}_{\infty}$ *is ω-inconsistent.*

Proof: A_0 is designed to "say" 'not all of the A_i are true, for $i > 0$' Assuming this is how A_0 is to be interpreted, we could reason like this. If A_0 were true, then each A_i would be true too by construction. So, A_0 must be untrue. But then, some A_n would be positively false (iterated fusions do this), making A_0 true. A contradiction.

This means that something must go wrong with the interpretation of A_0, on pain of inconsistency. The universal quantifier must pick out more than just the standard numbers. The argument goes like this, formally.

Consider the value $I(A_0)$ of A_0 in some model. If $I(A_0) \neq 0$ then $I(A_0^{n+1}) = I(A_{n+1}) = 1$ for some n (by iterated fusion). This means that $I(T\ulcorner A_{n+1} \urcorner) = 1$ by the T-scheme, and hence $I\big(\exists y \big(R(\underline{n}+1, y) \wedge Ty\big)\big) = 1$ by the definition of R. Hence, $I(A_0) = 0$, contradicting what we assumed.

So, we must have $I(A_0) = 0$, and hence $I(A_0^{n+1}) = I(A_{n+1}) = 0$ for each n. This means that $I\big(\exists y \big(R(\underline{n}+1, y) \wedge Ty\big)\big) = 0$ for each n. In other words, $\mathbf{Ł}^{\#\,T}_{\infty} \vDash \exists y \big(R(\underline{n} + 1, y) \wedge Ty\big)$ for each n, but $\mathbf{Ł}^{\#\,T}_{\infty} \vDash {\sim}\forall x \exists y \big(R(x+1, y) \wedge Ty\big)$, giving ω-inconsistency. ◁

5. Interpretation

If we wish to continue the search for a way to keep both truth and self-reference, we must weaken the catalogue of logical principles further than simply $\mathbf{Ł}_{\infty}$. It is clear from the proof that the fact that for any untruth, a finite number of fusions is sufficient to generate falsehood is to blame. This is reflected in the validity of Hay's rule (from ${\sim}(A^n) \to A$ for each n to derive A) in $\mathbf{Ł}_{\infty}$. The question arises: are there logics in the vicinity of $\mathbf{Ł}_{\infty}$ which do not contain this rule? And there are. The next weakest logic on the landscape is the substructural logic **CK**.[4] It has a smoother proof theory than Ł_∞, and it rejects all analogues of Hay's rule. If arithmetic, truth and self-reference are to live together, logics like **CK** are the place to make the attempt.[5]

Notes

[1]Specifically, we need just that $\vdash Prv\ulcorner A\urcorner$ iff $\vdash A$, $\vdash Prv\ulcorner A \to B\urcorner \to (Prv\ulcorner A\urcorner \to Prv\ulcorner B\urcorner)$, and $\vdash Prv\ulcorner A\urcorner \to Prv\ulcorner Prv\ulcorner A\urcorner\urcorner$. So, Prv contains as its provable extension all of the provable sentences in arithmetic, and it interacts with implication in the usual way. Note that these are quite broad conditions, as a truth predicate would do just as well as a proper provability predicate.

[2]Of course, it is also possible to either *ignore* conditionals, so that the problems like Curry paradoxes and Löb's theorem do not arise (this approach is taken by those who favour using a simple three-valued logic, such as Kripke [1] or Maddy [3]) or to accept *reductio* while bearing the contradictions that ensue (this approach is taken by those who favour the use of paraconsistent logic to deal with the paradoxes, such as Priest [4]). Neither of these analyses are at issue in this paper, interesting though they are.

[3]It was an insight of Ben Robinson that pointed me in the direction of defining a series of formulas with this general structure. I'm very greatful to him for the idea.

[4]See Slaney's "A General Logic" [5] for an introduction to the proof theory of **CK** and logics like it. The implicational fragment of **CK** is *very* natural proof theoretically. It arises from the Gentzen calculus for classical logic by banning contraction. It also arises from the λ-calculus. Just as intuitionistic implicational theorems are exactly the types of closed λ-terms, and relevant implicational theorems are exactly the types of closed λ-terms in which every λ-abstraction binds *at least one* free variable, the **CK** implicational theorems are exactly the types of closed λ-terms in which every abstraction binds *at most one* free variable.

[5]Thanks are due to Uwe Petersen and Graham Priest. Their comments helped make this paper clearer than it otherwise would have been.

References

KRIPKE, S.

[1] "Outline of a Theory of Truth," *Journal of Philosophy* **72** (1975), 690–716.

LÖB, M. H.

[2] "Solution of a Problem of Leon Henkin," *Journal of Symbolic Logic* **20** (1955) 115–118.

MADDY, P.

[3] "Proper Classes," *Journal of Symbolic Logic* **48** (1983), 113–139.

PRIEST, G.

[4] *In Contradiction: A Study of the Transconsistent*, Martinus Nijhoff, 1987.

SLANEY, J.

[5] "A General Logic," *Australasian Journal of Philosophy* **68** (1990), 74–88.

WHITE, R. B.

[6] "The Consistency of the Axiom of Comprehension in the Infinite-Valued Predicate Logic of Łukasiewicz," *Journal of Philosophical Logic* **8** (1979) 509–534.