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LOGICAL LAWS

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Abstract This paper is an introductory essay on the notion of a "Logical Law." In it, I show that there are three important different questions one can ask about logical laws. Firstly, what it *means* to be a logical law. Secondly, what *makes* something a logical law, and thirdly, what *are* the logical laws. Each of these questions are answered differently by different people. I sketch the important differences in views, and point the way ahead for logical research. (This paper is to appear in Routledge's forthcoming *Encyclopedia of Philosophy.*)

1 Introduction

There are at least three different kinds of answer to the question "what is a logical law?" One establishes what it means *for something to be a logical law. This answers the* semantic question*: What is the meaning of "logical law"? The second explains what* makes *something a logical law. This answers the* metaphysical question*: What is the ground of logical law? The third tells you what the logical laws* are*. This answers the* question of extension*: What is the extension of "logical law"?*

Even though logic is often seen as a completed science, the answers to all three questions are disputed. For example, there are at least three different conceptions of what it means for something to be a law of logic. Different conceptions account for logic in terms of necessity, truth in all models, and proof.

There are also different answers to the metaphysical question. If truth preservation is central to logic, then the ground of logic depends on the metaphysics of truth. If logic is a matter of the meanings of terms, then the metaphysics of meaning is important for logic. Unfortunately, there is no widespread agreement on the metaphysics of meaning or truth.

Finally, there is no widespread agreement as to what the logical laws are. There are two general disputes here. Firstly, it is not clear what notions count as logical. Does logic contain laws about identity, second order quantification, modality, or identity? Secondly, given agreement on the scope of logic, there are still questions about the logical laws in that area. Intuitionists, quantum logicians, relevant and paraconsistent logicians each reject things taken as laws by others, even in the language of "and," "or," and "not."

2 The semantic question

When we claim that something is a logical law, what do we mean? Sometimes we use "logical law" to mean "theorem of a formal system." Depending on taste, we might call theorems of first order classical logic, or some other formal system, laws of logic. This is a derivative use of the phrase, akin to calling the claims of a particular scientific theory laws of nature. The propositions of a theory may be laws of nature, but only if the theory 'gets it right.' The theory must describe the world in the right way for its claims to be laws of nature. Similarly, theorems of a formal system are only logical laws if they 'get it right' in some appropriate way. The interest consists in elaborating what it means for a theory to 'get it right.' So, by "logical law" we do not simply mean "theorem in a particular formal system."

Another non-answer to this question is that logical laws are the ways we cannot help but think. Anyone familiar with the history of logic in the twentieth century will be aware that almost any principle that some take to be a logical law, others reject as invalid. If a logical law is something to which no-one can help but assent, nothing counts as a logical law. No purely psychological answer, in terms of "laws of thought," will give us logical laws.

Now to a more plausible account. The goal in the study of logic is an account of deductively valid inferences: these are the logical laws. A deductively valid inference is one in which necessarily, if the premises are true, so is the conclusion. Laws of logic are those inferences that are *necessarily truth preserving*. This conception goes back to Aristotle.

> A syllogism is a form of words in which when certain assumptions are made, something other than what has been assumed necessarily follows from the fact that the assumptions are such. *Prior Analytics*, 24b18

We are interested in necessarily truth preserving inferences because given these, and given true premises from which to infer, we will never (and in fact, *can* never) deduce falsehood. Given our concern for finding truth and avoiding falsehood, it is easy to see why logical laws are interesting, on this picture. It is also clear how logic can be normative. Given that we have the goal of gaining truth and avoiding falsehood, we *ought* to deduce validly. (Which is not to say that we ought not reason in other ways as well.)

We have one answer to our semantic question: A logical law is a necessarily truth preserving inference. However, many have found this answer unsatisfactory. The main reason for dissatisfaction with this account is the reliance upon the intensional notion of necessity. Some have sought to give an account of logical laws that has no recourse to necessity, or to any other intensional notion.

There are two important analyses of logical consequence which, at least at first glance, avoid using the notion of necessity. One analysis was given its canonical exposition by Tarski in his essay "On the Concept of Logical Consequence" (1936) though it was prefigured by Bolzano in 1837. Tarski wrote

> The sentence X *follows logically* from the sentences of the class K if and only if every model of the class K is also a model of the sentence X.
>
> (Tarski 1936, page 417, emphasis his)

There are no modal notions here, provided that there is an adequate non-modal explanation of what it is to be a model. There are a number of ways to do this, and there is no need for us to consider them in detail. Suffice to say that models are structures in which each of the non-logical constants in a language is given an interpretation in terms of the objects in the model, and the logical constants have a fixed interpretation in each model.

Model theoretic notions of consequence do not, as they stand, make any appeal to modality. However, they do not give a modal-free account of necessary truth preservation.

For example, the model theoretic account needs a collection of logical terms. If *all* terms are logical, then the interpretation of each term is fixed, and there is only one model. This would make logical truth vacuous, as Tarski realised (1936). Now suppose that the terms "blue" and "coloured" are not counted as logical terms. The deduction "My car is blue; therefore my car is coloured" fails to be valid on the model theoretic account, because we can assign disjoint extensions to the predicates "blue" and "coloured." In contrast, the inference is necessarily truth preserving. In this way, necessary truth preservation and model theoretic validity come apart. (For a detailed account of how the notions differ, see Etchemendy's *The Concept of Logical Consequence* (1989).)

These two approaches to analysing logical validity differ quite sharply from a third account. According to the proof theoretical account of validity, an argument is valid just when there is a proof of the conclusion from the premises. This approach is championed by Prawitz (1974) with many other constructivists, and Wagner (1987) (a non-constructivist) who sees this view in Frege. The proof theoretical account grounds validity in the meanings of the logical constants. These meanings determine the validity of simple deductions (such as conjunction elimination: from $p \wedge q$ to derive q). An argument is valid if and only if there are simple deductions from the premises to the conclusion.

On this account, logical validity is analytic, and the epistemic function of logic (as a calculus for ideal justification) is obvious. As with model theory, a proof theoretical account of validity depends on the choice of logical particles, and the rules that govern them.

The proof theoretical account differs from the other two approaches. If validity is ultimately proof theoretic, then the class of logical validities is recursively enumerable. If we can list the basic rules, we can list the valid deductions (provided that proofs are finite). Similarly, validity is compact (if K follows from X, then K must follow from some finite subset of X). Neither of these properties is essential to the other conceptions of logical validity.

Without doubt, each of these conceptions has a place in the study of logical law. It is harder to discern how they are to be related. My tentative proposal is this. Validity is a matter of necessary truth preservation. Model theoretic validity is important, because models can represent possibilities (however ontological commitment to possibilities is to be analysed). Sometimes, models do not represent real possibilities (as when the extension of "blue" is not a subset of the extension of "coloured") so, model theoretic validity can differ from actual validity. Similarly, proof theoretic validity can coincide with acutal validity. Although analytic truth and necessary truth may not coincide in all cases, simple deductions involving logical particles are instances of necessary truth preserving inferences. Perhaps these simple deductions will capture all of the necessary truth preserving inferences in the language. In this case, the proof procedure is complete. On the other hand, the validities in the language may not be recursively enumerable, as in the case of second order logic. In these cases, no recursive proof theoretical account will capture all of the validities.

3 The metaphysical question

Even if we have decided what we mean when we call something a logical law, the issue of the *ground* of logical law remains. If validity is necessary truth preservation, an account has to be given of necessity and of truth. Similarly for the model theoretic approach. If models are to bear any resemblance to the world, truth-in-a-model ought to have the same structure as truth simpliciter. In these cases, doctrines of truth are relevant to the outcome of logical theory. For example, intuitionists have claimed that at least in the domain of mathematics, truth comes by way of construction or verification. This claim has resulted in disagreement about logical laws. The case is similar with quantum logic. Given a particular reading of the correspondence theory of truth (in which the facts to which propositions correspond are modelled by subspaces of Hilbert spaces is quantum physics), quantum logic seems to follow naturally. On the other hand, the correspondence, coherence or pragmatist theories of truth do not seem to dictate a particular theory of logic. Metaphysical doctrines of truth are relevant to an account of logic, but they need not determine that account.

Some take it that logical validity is independent of any particular account of truth, because logical laws are purely a matter of convention. Clearly convention has a large part to play in language, but it is much more than this to say that logical laws are simply true by convention. For example, it is clear that it is a matter of convention that "snow is white or snow is not white" is used to express a truth (and hence, a logical truth). It is also a matter of convention that "snow is white" refers to snow. In this case it would be very strange to say that the whiteness of snow is *purely* a matter of convention. Similarly, "snow is white or snow is not white" is not true solely by convention, but also by the way the world is.

Convention in logic is more at home in the view of logic as proof theory. Here, validity is a matter of the meanings of logical particles. Terms get meanings by convention. However, this does not exempt us from the difficult matters of semantics. Logic depends on the meanings of individual particles like "not." Is the meaning of "not" a matter of its truth functionality, and does this yield classical negation? Or is negation to be analysed inferentially, with intuitionists? The analysis of meaning undergirds the proof theoretical account of validity. No account of logical validity exempts the practitioner from the difficult task of giving an account of the ground of logical law. This is essentially a matter of metaphysics and semantics.

4 The question of extension

Even given answers to the first two questions, we are not at an end. We have yet to give an account of which inferences qualify as logical truths. Logicians have a ready supply of formal systems designed to do just this. These systems differ from each other in a number of ways. They come with a range of different languages. They can contain the truth functional connectives, quantifiers for objects (both the standard existential and universal quantifiers, and perhaps more exotic quantifiers) quantifiers of higher order, the identity relation, modalities of various sorts and so on. There is a whole wealth of different formal systems designed to give an account of valid inference in different languages. If validity is construed as necessary truth preservation, there is no need to make a principled choice of logical constants, because any term may be relevant in determining validities. The only choice is a pragmatic one, of which laws are worth studying. The way is open for a logic of colour, a logic of perception sentences, a logic of action, or a logic of all three. These 'logics' will be partial accounts of validity, just as a theory of electromagnetism is a partial account of physical law.

On the other accounts of validity, a distinction must be made between logical and non-logical terms, in order for there to be a univocal sense of logical validity. This is not an easy task, because there are no principled criteria that something must satisfy in order for it to be a logical constant. People disagree over whether generalised quantifiers of objects (such as "uncountably many") and second order quantifiers ought to feature in logic. (For a helpful discussion of generalised quantifiers, see Sher (1991), and for second order logic, see Boolos (1975), Shapiro (1991) and the literature cited there.)

Secondly, formal systems may differ in terms of their results, even if they have the same language. For example, intuitionistic logic and classical logic disagree about double negation elimination (not not p; therefore p) provided that you take intuitionistic logic and classical logic to have the same vocabulary. There is a subtlety here. It is obvious that two different formal systems can be reconciled by treating them as modelling different things. Then intuitionistic logic and classical logic can be seen as not disagreeing, because they have different negations. Quine said as much when he claimed that changing a logic amounted to changing the subject (1970).

However, this is not the whole story. Classical logic and intuitionistic logic can disagree as to the validity of real arguments. Given a particular argument in natural language, an intuitionist and a classical logician may disagree about its validity. If there is a fact of the matter as to whether the argument is valid, then classical logic and intuitionistic logic can be seen to disagree about this fact. The change of logic does not involve a change of subject when the subject (in this case, natural language arguments) is fixed in advance. Given a particular instance of the natural language "not," classical logic and intuitionistic logic are different accounts of the valid arguments involving this fragment of natural language. The natural language particle is *prior* to its formalisations, which are intended to capture the meaning of the particle. These formalisations can capture the meaning well, or not well, and hence, these formalisations can disagree.

With intuitionistic logic, the locus of disagreement is the conception of truth, or alternatively, the meanings of the logical particles. Given a verificationist or constructivist view of truth, you have a reason to be an intuitionist with regard to the standard logical particles. However, if you disagree with constructivism, and you do not analyse the meanings of logical terms in constructivist terms, you can use intuitionistic logic as a logic of necessary provability preservation. Disagreement about intuitionism can only be resolved by agreeing on theories of truth and meaning. The case is similar with quantum logic, as we saw before.

Disagreement becomes more subtle when we consider paraconsistency and relevance. According to most logics, the argument from $p \wedge \sim p$ to q is valid, but according to paraconsistent logics (including all logics of relevance (Anderson, Belnap and Dunn 1975 and 1992), and the literature cited there) it is not. In this disagreement no doctrines of truth or meaning stand out as motivating the peculiar features of the logics. Instead, there seem to be two major reasons that proponents of these systems can put have for analysing validity in this way.

The first is quite simple. If a contradiction were true, or even possibly true, it would be clear that the inference from $p \wedge \sim p$ to q would be invalid. While many doubt the coherence of this supposition, enough work has been done in the field of paraconsistent logic to show that the approach is coherent (Priest, Routley and Norman, 1989). If we admit the need to be able to deduce in inconsistent situations (whether they are epistemic situations, possible situations, or actual situations) a logic without the inference from $p \wedge \sim p$ to q is necessary. Note that this does not involve any particular doctrine about the nature of truth (correspondence, pragmatic, or coherence), but simply, the view that it is possible that a contradiction be true. Resolving disagreements like this is a matter of examining the arguments for and against the truth of contradictions.

There second motivation for the view that the argument from $p \wedge \sim p$ to q is invalid is quite different. On this view you need not hold that it is possible that a contradiction be true. Instead, you maintain that for an argument to be valid, and for a conditional to be true, its antecedent must be relevant to its consequent. If this is the case, then there may be a reason to reject the inference. Again, this disagreement is not about a particular doctrine of the nature of truth. Instead, it is a disagreement about the relationship between relevance on the one hand, and validity and conditionality on the other. Opponents can trade intuitions about the validity or otherwise of individual arguments, but this kind of discussion is rarely fruitful. A saner approach might go like this: if the relevantist can develop a coherent theory of validity, which models our valid argument at least as well as can be done otherwise, and which in addition, gives an account of phenomena that cannot be modelled otherwise, then clearly, the relevantist position is viable and valuable. If it cannot, it will not succeed as a theory of validity.

In all of these considerations, we have seen that logic is quite similar to other sciences. The practitioners have some idea of what the subject matter is (valid inferences) but there is debate about its exact nature. The task is to model the subject matter, and there is no wide consensus about how this is to be done. However, as in other sciences, this does not mean that the study is not worthwhile, or that it will not enrich our knowledge of the way things are.

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