On Logics Without Contraction

January 1994

Greg Restall

A thesis submitted to the Department of Philosophy, University of Queensland,
in fulfillment of the requirements for the degree of Doctor of Philosophy (Ph.D.) in philosophy.

God cannot better be served,
than if by law
ye restrain this unlawful contracting.

— Archbishop EDWIN SANDYS [170]

Declaration

I declare that the work presented in this dissertation
is, to the best of my knowledge, original
and that the material has not been submitted, in whole or in part,
for the award of any other degree at this or any other university.

Contents

Acknowledgements

What would you think if I sang out of tune,
Would you stand up and walk out on me.
Lend me your ears and I'll sing you a song,
And I'll try not to sing out of key.
I get by with a little help from my friends.

— JOHN LENNON and PAUL McCARTNEY
'With a Little Help from My Friends' [67]

There are *many* people who deserve thanks for their contribution to my ideas. None of them will agree with everything asserted here, but that matters little. This thesis was written for the Doctoral programme in the University of Queensland, and I spent a very enjoyable time participating in the life of the Philosophy Department there. My supervisor was Graham Priest, who introduced me to the pleasures of deviant logic, and whose logical acumen, insights, and philosophical ability have been willingly shared. Without his encouragement, this thesis would not have existed — it was a joy to work with him. The rest of the Department were also willing listeners to my often crazy ideas. I would especially like to thank André Gallios, Ian Hinckfuss, Gary Malinas and Ellen Watson for thought provoking discussion. Among fellow students, discussions about the nature of logic with Greg Littmann, Daniel Nolan, Mark Patterson and Michelle Styles-Williams were stimulating and helpful. In Semester 2 of 1993, I had the opportunity to teach some of this material to a third-year logic class in the Philosophy Department. Their perceptive responses and questions were very thought provoking, and I thank them for that. I can only hope that they learned as much from me as I did from them. Grace Dunn, our departmental secretary was a God-send in many different ways, from helping me get through the administrative details of being a postgraduate student and a tutorial assistant, to booking flights around the country. Sharing an office with Inari Thiel also made working on this thesis much more enjoyable than it otherwise would have been. Steven Young performed the laborious task of reading through the manuscript to find infelicities of expression. It is a joy to have friends like these.

Working with the rest of the Australian Logic community has been very enjoyable. The influence of John Slaney on this thesis is quite obvious — he answered many of my questions, and gave me more questions in return. His assistance was helpful, and I owe a great debt to his work in this field. I've also benefitted from conversations with Ross Brady, Allen Hazen, Dominic Hyde, Ed Mares, Chris Mortensen, Bob Meyer, Uwe Petersen and Richard Sylvan. These people helped me get my way around what turned out to be a wide and deep reservoir of work in this area.

Further afield, the work of Geoff Keene in establishing the *Relevant Logic Mailing List* helped me keep in touch with other deviant logicians in other parts of the world. In addition, the wonders of electronic mail have given me the opportunity to share ideas with many in associated areas. Electronic communication with Jon Barwise, Chris Brink, Mark Lance, Vaughn Pratt, Ingrid Rewitzkey and Mikulas Szabolcs proved helpful. Extended discussions with J. Michael Dunn and Chris Menzel helped me to get a number of issues clearer.

I would like to thank two people who have kept me human during this period. Without encouragement from my mother, over the many years of my education, I would never have got to this stage. Finally, Christine Parker. Although we haven't talked about logic very much, her example, her enthusiasm, and her love mean more to me than I can express in words.

As much as each person mentioned here has influenced the product you have before you, none had such an effect that they are responsible for the errors. For those, only I am to blame.

The research reported in this thesis was conducted with the assistance of an *Australian Postgraduate Research Award* and a tutorial assistantship from the University of Queensland. Thanks are also due to the Centre for Information Sciences Research at the Australian National University, for funding enabling me to report some of this work there.

> . . . I should like to outline an image which is connected with the most profound intuitions which I always experience in the face of logistic. That image will perhaps shed more light on the true background of that discipline, at least in my case, than all discursive description could. Now, whenever I work even on the least significant logistic problem, for instance, when I search for the shortest axiom of the implicational propositional calculus I always have the impression that I am facing a powerful, most coherent and most resistant structure. I sense that structure as if it were a concrete, tangible object, made of the hardest metal, a hundred times stronger than steel and concrete. I cannot change anything in it; I do not create anything of my own will, but by strenuous work I discover in it ever new details and arrive at unshakable and eternal truths. Where is and what is that ideal structure? A believer would say that it is in God and is His thought. [72]

Introduction

> Was it just a dream? Were you so confused?
> Was it just a giant leap of logic?
> Was it the time of year, that makes a state of fear?
> Methods, were there motives for the action?
> — ROB HIRST 'My Country' [56]

It's easy to prove everything if you're not too careful. For example, let q be a claim, like the claim that Queensland has won the Sheffield Shield. It's false, but it would be nice if it were true. We can prove it like this. Consider the sentences

$$\text{If } (\gamma) \text{ is true, then } q \text{ is true.} \tag{γ}$$

and

$$\{x : x \in x \to q\} \in \{x : x \in x \to q\}$$

Given the way we use truth or set membership, these claims appear to be propositions p which satisfy $p \leftrightarrow (p \to q)$. If (γ) is true then, if (γ) is true, so is q (because that's what (γ) says). So, we have $p \to (p \to q)$. Conversely, if we have 'if (γ) is true then q is true' we have (γ), because again, that's what (γ) says. So, (γ) appears to be equivalent to 'if (γ) then q is true.' The same sort of reasoning holds for the other sentence. However, from $p \leftrightarrow (p \to q)$ we can prove q very simply. Here is a proof written in tree form:

$$\dfrac{\dfrac{\dfrac{\dfrac{p \leftrightarrow (p \to q)}{p \to (p \to q)}}{p \to q}\,[!] \qquad \dfrac{p \leftrightarrow (p \to q)}{(p \to q) \to p}}{p} \qquad \dfrac{\dfrac{p \leftrightarrow (p \to q)}{p \to (p \to q)}}{p \to q}\,[!]}{q}$$

So, our cricketers should stop training to win the elusive Shield. We have *proved* that we have already won.

Unfortunately not. This doesn't count as a proof because it 'proves' too much. If it were valid, we could prove anything. The sorry thing is that logicians are not agreed as to where proofs like these break down. Most take it that the claims *don't* really satisfy $p \leftrightarrow (p \to q)$ because set theory and semantics don't really work in the way we thought they did. The paradoxical deduction is taken to be a proof that our propositions p didn't really satisfy $p \leftrightarrow (p \to q)$ in the first place. For set theory, the verdict is almost unanimous. It is said that there is no set $\{x : x \in x \to q\}$. For truth the matter is more difficult. According to the common wisdom, either (γ) isn't (or doesn't express) a proposition, or if it is (does) it's not the same claim as the claim that if (γ) is true, then q.

But some don't stop the deductions there. A brave few allow such propositions and sets but blame a step in the deduction we saw above. Granted that $\to$ is an implication (so it satisfies *modus ponens*) the only likely place to stop the deduction is at the steps marked [!] — an inference called *contraction* (also called **WI** in what follows). Only a brave few have thought that an inference like contraction can be faulted and this is why many have sought to cast doubt on intuitive set theoretic and semantic principles. However, there is rhyme and reason behind rejecting inferences like contraction. To see this, consider another inference which also features in paradoxical deductions. We will call propositions of the form $p \wedge (p \to q) \to q$, *pseudo modus ponens*. From an instance of *pseudo modus ponens* and our propositions p we can derive q again. Without using contraction. If $p \leftrightarrow (p \to q)$ we easily get

$$p \wedge p \to q$$

from *pseudo modus ponens* by substituting p for $p \to q$. But supposing that $p \wedge p$ is equivalent to p (a sensible thing to suppose) we have

$$p \to q.$$

But this and our $p \leftrightarrow (p \to q)$ gives p, from which we can deduce

$$q.$$

So we're back where we were before. The other principles we used were sound, so if we accept naïve semantics and set theory, we must reject *pseudo modus ponens* too. Before recoiling in horror like many very good logicians have done before you at the thought of such a thing [95], take a moment to look at the similarity between contraction and *pseudo modus ponens*.

For example, to demonstrate the inference of contraction (**WI**) in a simple natural deduction formalism we have to use a proof like this one.

$$\dfrac{\dfrac{[p] \qquad \dfrac{[p] \qquad p \to (p \to q)}{p \to q}}{q}}{p \to q}\,[!]$$

In the proof, each node is a simple logical consequence of those immediately above it (by *modus ponens*) except for the step marked [!] at which conditional proof is applied, discharging *both* bracketed assumptions. Now consider a proof of *pseudo modus ponens*.

$$\dfrac{\dfrac{\dfrac{[p \wedge (p \to q)]}{p} \qquad \dfrac{[p \wedge (p \to q)]}{p \to q}}{q}}{p \wedge (p \to q) \to q}\,[!]$$

Now the moves are eliminating conjunctions, as well as *modus ponens* and conditional proof. Again, we must discharge two instances of an assumption at once. If we do without that, we get rid of contraction. A logic which is strict about repetitions of assumptions in just this way is called a *contraction-free logic*, and these logics are the topic of this thesis.

In a contraction-free logic, there is no way (in general) to infer $p \to q$ from $p \to (p \to q)$, or to prove $p \wedge (p \to q) \to q$. For the latter, the most that could be proved is something like

$$p \wedge (p \to q) \to (p \wedge (p \to q) \to q)$$

which clearly records that the antecedent $p \wedge (p \to q)$ had to be used twice to deduce the consequent q. This is the real axiom form of *modus ponens*. The *pseudo* variety is the imposter.

These reflections are not new. In 1942 Curry showed that paradoxes like Russell's and the liar could be formulated without recourse to negation [25]. These are the kinds of paradoxes we have considered. So, these forms of the paradoxes bear his name — they are *Curry paradoxes*. Then others noticed this too, and some concluded that contraction-free logics were the best way out of trouble. Brady [172], Curry [25], Geach [47], Moh [97], Myhill [101,102] and Prior [119] each had something good to say about contraction-free logics. But study in the area has not been easy, and it has not progressed very far. In this thesis, I attempt to make more progress along this front, by studying logics without contraction.

In the early chapters I set the scene. In 'Logics and LOGIC' I argue against those that take classical logic as primary, in order to set the stage for non-classical logics like those we intend to study. Then in the next chapters 'Basic Proof Theory,' 'Propositional Structures,' and 'Relational Structures,' I define the logics we will consider, I prove simple properties by basic proof theoretic means, and then I define a simple algebraic semantics for the propositional fragments of the logics, and then their quantified counterparts. None of this is particularly difficult, and the major original results are in 'Relational Structures' in which I generalise standard classical model-theoretic results to a non-classical setting.

Then we move to more interesting semantic structures for contraction-free logics. In 'Simplified Semantics 1' and '2' I generalise some results of Priest and Sylvan [118] to give contraction-free logics a ternary relational semantics. This will prove useful later, because these structures turn out to have independent interpretations. Before going on to these interpretations, I take a detour into 'Decidability.' In that chapter I prove that all of the logics we consider are decidable (in their propositional fragments). This proceeds by way of a Gentzen system for the logics. This work relies heavily on earlier work by Dunn [28], and independently, Minc [171], then taken up by Meyer [85], Giambrone [48] and Brady [18,19]. I generalise it even further to give decidability results for a number of systems for which decidability was previously unknown. Then we go on to consider applications of the logics. In 'Contraction-Free Applications,' we look at vagueness, commitment, and actions; each giving us an interpretation for contraction-free ternary relational semantics. Then Barwise's recent work on conditionals in situation semantics give us another interpretation for the ternary relational semantics for contraction-free logics. We explore this in 'Information Flow.' Then we round off that section by showing that one of our systems is the natural extension of the Lambek associative calculus, and that it can be used to model interesting linguistic phenomena.

From there we go back to paradoxes. We look at them in the context of 'Contraction-Free Arithmetic,' to examine how truth and provability can function in a contraction-free setting, and then at 'Naïve Comprehension.' These chapters contain new results that pave the way for further research in the areas.

In the penultimate two chapters we examine two ways of retrieving contraction in our logics. The first arises from finitely valued logics, and the second by introducing a special modality into the logic. Both of these give us a clearer picture on what it means for a logic to lack contraction.

Finally, we look back on what we have learned, and 'check the score' against rival non-classical logics and other approaches to the paradoxes. From all of this, we are able to get more of an idea of the behaviour of contraction-free logics, and in so doing we will fulfil more of the expectations of logicians of yesteryear who wondered whether contraction was something we could do without.

EXCURSUS: There are a number of conventions used in the writing of this thesis — knowing them may be helpful to the reader. Firstly, an 'EXCURSUS' like this is a comment that lies a little off the beaten track of the thesis as a whole. It is pertinent to that point in the discussion, but following it fully would lead us far away from the topic at hand. Feel free to read or ignore each EXCURSUS as you find it. (They end with a '□' in the right margin.)

The policy on 'I' and 'we' is simple. Whenever something pertains to just the author, I use 'I.' 'We' is reserved to refer to both you and me together. I will write things like 'we will see that ...' meaning that as you continue to read and I continue to write, we will come see that very thing. For contentious claims, I may write 'I think ...' and leave you out of it, because you may not agree. But on the other hand, I may write 'we think ...' if I am relatively confident that you will agree.

Proofs start with a '*Proof:*' and end with a little '◁' in the right margin. It is customary to find in works like this that some proofs are just too tedious to write out completely. In cases like that I have taken the liberty to sketch the method of the proof and leave the rest to you. I often call this an 'exercise left for the diligent reader.' Let me assure you that I *have* completed the details and I have only left them out because a 280 page thesis with a few exercises is better than a dull and tedious thesis with 600 pages.

Finally, endnotes are indicated by a raised number, like this one.[1] Endnotes are to be found in the 'Notes' section at the end of each chapter. □

Chapter 1
Logics and LOGIC

Letting a hundred flowers blossom
and a hundred schools of thought contend
is the policy for promoting the progress
of the arts and the sciences.

— MAO TSE-TUNG
On the Correct Handling of Contradictions Among the People [76]

As we saw in the introduction, paradoxes draw our attention to logics without contraction. But this raises an important question. Is this approach coherent? Why not try to give an account of the paradoxes while keeping classical logic as the account of valid reasoning? In this chapter we will see some contemporary arguments that are intended to prove the primacy of classical logic. In dealing with these arguments, we will have the beginnings of an apologetic for deviant logics in general, and contraction-free logics in particular.

1.1 Paradoxes

The paradoxes of self-reference have provided one of the driving forces behind much of the semantic and set theoretic enterprise of the twentieth century. To the paradoxes we owe the type hierarchy of Russell and Whitehead, Zermelo-Frænkel set theory, and the Tarskian hierarchy of languages and truth predicates. These theories in semantics and set theory are shaped by their particular responses to the paradoxes. Paradoxes provide the data that prospective theories (of truth, of sets, and so on) must deal with. This sentiment was expressed by Bertrand Russell [136].

> It is a wholesome plan, in thinking about logic, to stock the mind with as many puzzles as possible, since these serve much the same purpose as is served by experiments in physical science.

In this chapter I will attempt to show that the puzzles provided by the paradoxes of self-reference don't just give us material useful in formulating theories in semantics and set theory — they also give us good reasons to explore logical systems that deviate from classical logic in particular ways. The paradoxes give us reason to hold that classical logic is not a good candidate for modelling valid inference.

Many definitions of the term "paradox" have been proposed, and in our context nothing much hangs on the issue. However we would still do well to get the term clear. The definition we will work with is this:

> **Definition** A *paradox* is a seemingly valid argument, from seemingly true premises to a seemingly unacceptable conclusion.

So, such disparate things as the paradoxes of self-reference, the sorites paradox,[1] and the paradoxical decomposition of spheres are counted as paradoxes. For the moment, consider the paradoxical decomposition of spheres. The paradox is given by a rather technical argument in mathematics, where it is shown that it is a consequence of the axiom of choice that one solid sphere can be decomposed into a finite number of pieces, which can then be reconstructed into two solid spheres, each of the same volume as the first. A somewhat surprising conclusion.

This example shows that a paradox does not have to be an argument to a *contradiction*; any seemingly unacceptable conclusion will do. Another example might be provided by the paradoxes of material implication — which show that some possibly plausible assumptions lead to odd conclusions about true conditional statements.

Given such a conception of paradoxes, we can see that a paradox can elicit any of a number of responses. The three viable ones are to

(1) Explain why a premise is not true.
(2) Explain why the argument is not valid.
(3) Explain why the conclusion is acceptable.

All three approaches have been taken in the history of our dealings with paradox. For example, the paradox of the barber: Consider a small village. In this village lives a (male) barber. This barber shaves all and only those men in the village who don't shave themselves. Why does this give us a paradox? Ask yourself whether or not the barber shaves himself. If he doesn't shave himself, it follows from the condition given that he does shave himself. If he does shave himself, it follows by the condition that he doesn't. Given the plausible assumption that either he does shave himself or he doesn't, we get the unacceptable conclusion that he both does and doesn't shave himself. When presented with this paradox, the solution is simple. We perform a *modus tollens* to the effect that there cannot be a barber who satisfies this condition. The argument is most certainly valid, and the conclusion is most certainly false. We solve the paradox by rejecting a premise.

Both of the other approaches are taken for different formulations of the paradoxical decomposition of spheres. If we take the paradox to conclude that we can take apart one *physical* sphere and reassemble it to form two of the same size, we can say that the argument isn't valid. Inferring that this decomposition could be done with physical spheres from the fact about spheres in the abstract Euclidean space R^3 is simply invalid. The process involves cutting the sphere into what mathematicians call 'non-measurable' pieces, and this is impossible given that the spheres are made up of atoms. So, the nominated way to slice the sphere cannot be physically instantiated.

If we take the argument to conclude that this can be done in R^3 then (some) mathematicians will tell us that the conclusion is acceptable. This is done by reassuring us that this does not mean that the phenomenon has any physical instances.

These examples show that all three ways of dealing with paradoxes have their place. Let's move from general considerations to a specific paradox.

1.2 The Liar

The paradoxes of self-reference are a particular group of paradoxes that each combine the apparatus of self reference with notions from semantics or mathematics to give untoward conclusions. We will focus on the liar paradox, although what we have to say can be adapted to many other paradoxes in a relatively painless fashion. In particular, even though the paradox isn't *explicitly* related to contraction, it is central to any discussion of the paradoxes, and possible responses to the liar are very closely related to the possible responses to the 'Curried' forms of the paradoxes.

Now it is natural to hold that a theory of truth will give us

$$\vdash \mathsf{T}\langle p\rangle \leftrightarrow p$$

for all sentences p, where $\langle p\rangle$ is a name of the sentence p.[2] This condition makes sense, and it seems to capture at least some aspect of what we mean by "true." An assertion of "$\langle p\rangle$ is true" seems to have just the same inferential force as the assertion "p." The claim "Everything Bob Meyer said today is true" has just the same inferential force as asserting everything Bob said today.[3]

Unfortunately this assumption about truth leads to paradox, given a few reasonably plausible and well known steps. One culprit is the putative statement:

$$(\lambda) \text{ is not true} \tag{λ}$$

Given our condition on truth it seems quite easy to deduce a contradiction to the effect that (λ) is both true and not true. This much is well known. What to do with it is the problem.

1.3 Orthodox Paradox Solutions

There is almost universal agreement that no *obviously* correct account of the self referential paradoxes has yet surfaced. The central thrust of much of the current research in paradoxes is to see how the paradoxes can be dealt with inside the framework of classical logic. These solutions fall into two classes. The solutions in one class deny that the liar makes a statement. The others accept that we can state things using the liar, but that somehow this does not an unacceptable conclusion. In what follows, I sketch an account of the shortcomings of these approaches.

'Non Expressible' Solutions This class of solutions to the paradoxes attempt to formulate some reason why the paradoxical sentence cannot enter the realm of valid inference. So, they generally fall into the class of paradox solutions that seek to explain why a premise is not true: in this case, the premise to the effect that (λ) is (or expresses) a proposition. This is often attempted by way of a syntactic theory that counts the expression as not well-formed. As such, it gets off on the back foot, for the expression does seem to be a well-formed English sentence. However, some attempts are made, a familiar one uses the structure of type theory.

RAMIFIED TYPE THEORY The theory of types introduces the plausible conditions of *typing* on its formal language. In other words, different syntactic objects have different types, and these correspond to different bits of the world. Typically, a type theory will have a domain E of entities, a domain P of propositions, and other domains of properties of different orders. Each domain can be quantified over — for example, if Q is the domain of first order properties, we might have:

$$\vdash \forall x^E \forall y^E (x = y \supset \forall Z^Q (Zx \equiv Zy))$$

However, if we have a single domain P of propositions, we are stuck with a problem. There seems to be nothing preventing us from formulating a proposition

$$\forall x^P (x = p \supset \sim x)$$

which is identical to the proposition p itself. However, a quick inspection shows this to be a liar-like proposition. It states that any proposition identical to itself is false. Given plausible conditions on identity (namely that $p = p$ is true, and if $p = q$ and p is true, then so is q) and classical predicate calculus, we can reason as follows:

(1)	$\forall x^P (x = p \supset \sim x) \vee \sim \forall x^P (x = p \supset \sim x)$	Excluded Middle.
(2)	$(p = p \supset \sim p) \vee \sim \forall x^P (x = p \supset \sim x)$	Instantiating a quantifier.
(3)	$\sim p \vee \sim \forall x^P (x = p \supset \sim x)$	As $p = p$.
(4)	$\sim \forall x^P (x = p \supset \sim x)$	$\sim p$ is $\sim \forall x^P (x = p \supset \sim x)$.
(5)	$\exists x^P ((x = p) \wedge x)$	Classical Quantification moves.
(6)	p	By identity.
(7)	$p \wedge \sim p$	From 4 and 6.

Propositional quantification gives us all we need to prove the paradox, once we have the required proposition p. Denying that a proposition like p can ever exist is *ad hoc* unless it can be motivated by some other consideration. The structure of type theory gives us a way to do just that. The general principle is that the quantifier in a proposition of the form $\forall x^P A(x)$ does not range over all propositions, but some subclass of that range, *which does not include the proposition itself.* The intuition is that quantifiers have to range over completed, or definite, totalities. As Thomason explains, if we think of the interpretation of a universally quantified statement as involving a process of constructing the corresponding proposition, and somehow establishing the interpretation of all instances of the proposition, the process becomes circular, and not well-founded [161]. This is a problem. The way out is to keep books on quantifiers, and have increasing domains P_1, P_2, $P_3 \ldots$ of propositions. In this case, the substitution cannot work, and we have no problem with the paradox.

However, it is a case of 'out of the frying pan and into the fire,' as we lose many other advantages of the type-theoretic approach. I will merely sketch some of the familiar, yet damning objections. Firstly, ramifying the types leaves us no way to state logical laws, as we can't quantify over *all* propositions, but it is just this quantification over all propositions which helps us both understand type theory, and give it a semantics.

Secondly, it results in a general ban on self reference, which is overkill. Statements like "This is in English," and "Everything I say in this chapter is true" are unformalisable in the ramified type theory. Also, if I say "the next thing Bob Meyer says is true," this could be formalised by giving my quantifier a higher type than the next thing Bob says. So far it's OK. But if his next statement is "what Greg Restall just said is false," I have failed to make a statement as there's no way of assigning types consistently. Perhaps a type-theorist could bite the bullet at this point, and agree that we have not stated anything — after all it appears to be paradoxical if we assume that we have. However, this is not all that is excluded on this approach. If I say "Everything Bob says is funny," and he says "Something Greg says is OK," then *again* we cannot consistently assign types to these quantifiers. This is overkill. A theory is very odd when it claims that whether or not I make a statement depends on the past, present, *and future* actions of others.

This approach is in trouble, and it's not clear how it could be modified so that it will begin to be an adequate solution.

Hidden Variable Solutions Other approaches attempt to admit the liar as an authentic statement, but argue it has a different truth value or semantic status to another claim that (λ) is not true:

$$(\lambda) \text{ is not true} \qquad (\lambda')$$

This is done by way of making explicit some property that the sentences don't share. As the relevant property is not obvious to the naked eye, I've called these *hidden variable solutions.* In our typology of paradox solutions, these can be taken to be explaining why a premise is false, provided that we take the claim that (λ) is true if and only if it is not true as a premise of our argument. Alternatively, if we take there to be some *argument* to this conclusion, this approach will be seen as explaining why the argument is invalid.

Barwise & Etchemendy In their book *The Liar* [10], Barwise and Etchemendy give refreshingly different accounts of propositions, and their relationship to sentences. The account they favour, is the *Austinian* one, as it is inspired by J. L. Austin's work on truth [8]. On this account, a proposition is modelled by an object $\{s; T\}$, where s models a *situation* (a chunk of the world determined by the utterance and pragmatic features of linguistic practice) and T models a *situation type* (some kind of restriction on situations). On this account, s is a set of (things that model) states of affairs, and T is a (thing that models a) state of affairs. On this account, the proposition $\{s; T\}$ is true if and only if $T \in s$. That is, if the situation picked out is of the type specified.

On this picture, some of the states of affairs will be of the form $[Tr, p; 1]$ or $[Tr, p; 0]$, where p is a proposition. These are the states of affairs that obtain when p is true, and false, respectively. Actual states of affairs are *coherent*, in that if $[Tr, p; 1] \in s$ then p is true, and if $[Tr, p; 0] \in s$ then p is false. So, on this account, a liar proposition will be of the form:

$$\lambda = \{s; [Tr, \lambda; 0]\}$$

where s is some actual situation. By the coherence condition, if $[Tr, \lambda; 0] \in s$, λ is false, and so $[Tr, \lambda; 0] \notin s$. So λ must be false. However, it doesn't follow that λ is true, for the state of affairs $[\mathrm{Tr}, \lambda; 0]$ is not a part of the situation s. The state of affairs obtains, but the liar shows us that it cannot be a part of the situation that is talked about. It follows that the true claim λ', to the effect that λ is false talks about an *expanded* situation which includes the falsity of λ. So, the hidden variable is the situation, which differs from λ to λ'.

All in all, this is an interesting attempt at solving the paradox. It is coherent, and it retains a lot of our intuitions. However, it does not succeed. Firstly, there are no propositions about the whole world — each proposition is about a particular situation, which is only a proper part of the world, as situations are modelled by sets, and the whole world is a modelled proper class on their picture. (If there were a world situation w where for every true α, $[\mathrm{Tr}, \alpha; 1] \in w$, then the paradox would return by way of the proposition $\lambda_w = \{w; [\mathrm{Tr}, \lambda_w; 0]\}$.) This seems to be an artefact of the modelling, and unless it is given some justification, it will not be able to withstand the weight that is put on it. One such weight is the analysis of the liar. It is argued that the state of affairs of the falsity of the liar cannot be a part of the situation the liar describes — on pain of contradiction. This is a consistent approach, but it doesn't give us an independent *explanation*. It really seems that the liar is general, and not context-bound in the same way as "Claire has the three of clubs" obviously is. More explanation must be given if this is going to count as a *reason* for blocking the paradoxical inference. More work has to be done to give an account of what states of affairs feature given situations. *Prima facie* it seems that situations are T-closed. That is, it seems that if $\{s; T\}$ is true, then the state of affairs of $\{s; T\}$ being true is a part of the situation s. It is unclear that we need to go to a larger situation to explain what it true in s, yet this is what Barwise and Etchemendy's account requires. If a substantive theory of the paradoxes is to be given, some kind of principled explanation of the behaviour of truth in situations must be given. Situation semantics certainly provides an ingenious place to stop the derivation, but the analysis falls short of telling us why we should want to stop it there.

There are many more views, none of which is particularly hopeful. (Which doesn't mean work on them should be discouraged — each of these analyses draws distinctions that are important, and they have enriched our vocabulary of concepts. They call our attention to distinctions that we are often hasty to fudge.) We need to examine the alternatives to see whether or not they are any more promising. I think they are.

1.4 'Deviant' Approaches

To stop these paradoxical deductions at a propositional step, there are three possible positions. One is the move from "(λ) is true iff (λ) is not true" to "(λ) is true and (λ) is not true," another is the move from "(λ) is true and (λ) is not true" to an arbitrary proposition q, and the last is to deny the validity of *modus ponens*. In tree form the places are:

$$\dfrac{\dfrac{\dfrac{\dfrac{p \leftrightarrow \sim p}{p \to \sim p}}{\sim p}[\mathfrak{b}] \quad \dfrac{p \leftrightarrow \sim p}{\sim p \to p}}{p}[\mathfrak{c}] \qquad \dfrac{\dfrac{\dfrac{\dfrac{p \leftrightarrow \sim p}{p \to \sim p}}{\sim p}[\mathfrak{a}]}{p \to q}[\mathfrak{b}]}{}}{q}[\mathfrak{c}]$$

I will call the approaches that halt the derivations at the steps marked $[\mathfrak{a}]$ *paracomplete*. This step is taken to be invalid because the proposition p has some kind of 'defective value,' such that p implies $\sim p$, but without $\sim p$ as a consequence of this. This is commonly described in terms of p being neither true nor false.

Paraconsistent solutions will either deny the validity of the step marked $[\mathfrak{b}]$ or that marked $[\mathfrak{c}]$. The reason for denying $[\mathfrak{b}]$ is that the proposition p is both true and false, and so $\sim p$ ought not deliver us $p \to q$ for arbitrary q. This is the approach that *must* be taken for the paraconsistent reasoner if the conditional satisfies *modus ponens*, as this is the only rule left. Denying step $[\mathfrak{c}]$ is the last resort for a paraconsistent solution to the paradoxes. Not much will be said about this kind of paraconsistency, for it seems that even if some conditionals (such as the material conditional) don't satisfy *modus ponens*, we can cook one up as follows: $p \Rightarrow q$ is shorthand for 'the argument from p to q is valid.' It would be *very* odd to hold that this didn't satisfy *modus ponens*. Our observations in Section 1.2 lead us to hold that p and $T\langle p\rangle$ have the same inferential force, so the inference from one to the other is valid, so we ought to have $p \Rightarrow T\langle p\rangle$ and *vice versa*. Whatever we think of other varieties of paraconsistency, the species that rejects this kind of detachment is an alternative we ought only consider as a last resort.

Paracomplete solutions have been advocated by Saul Kripke[4] [63] and Penelope Maddy [75] among others. Paraconsistent solutions have been put forward by Graham Priest and Richard Sylvan (Routley) [113, 116, 117]and Albert Visser [173]. These solutions to the paradox can be seen as explaining how the argument is invalid, or in the paraconsistent case as explaining how the conclusion (that (λ) is both true and not true) is not as bad as we might have thought.

1.5 Three Objections

Is there any *prima facie* reason against such a solution as either of these? If there is, it can probably be brought to bear on the project of contraction-freedom too. So, it is worth looking for possible objections. The literature provides us with a few candidates. Vann McGee in his *Truth, Vagueness and Paradox* [82][5] gives three objections to a particular paracomplete solution due to Kripke and others. In brief the three objections are:

The difficulty of learning the 3-valued logic. Here the difficulty is mainly a *practical* one, with regard to how hard it would be to reason in the 3-valued logic. McGee writes:

> The first obstacle is simply how difficult it would be, in practice, for us to use the 3-valued logic in place of the familiar logic. Classical logic has served us well since the earliest childhood, yet we are asked to abjure it in favour of a new logic in which many familiar and hitherto unproblematic modes of inference are forbidden. (page 100)

McGee responds to his own objection by showing how a particular formulation of the 3-valued logic can be given, using rules of proof very similar to a classical natural deduction system. In fact, there is only one difference between the two systems, which is not difficult to learn. However, McGee's objection is directed against any 'deviant' approach to the paradoxes, and ought to be considered.

The unavailability of scientific generalisations. This objection hinges upon a feature of the 3-valued logic used in the solution McGee criticises. The feature is that if p and q are evaluated as 'neither,' then so will the conditional $p \to q$. This is problematic in certain contexts. McGee writes:

> Consider Jocko. Jocko is a tiny fictional creature that lives right on the border between animals and plants. Jocko has many of the features we regard as characteristic of animals and many features we regard as characteristic of plants. Jocko's animallike characteristics are those we expect to find in protozoa, so that Jocko is also on the border between protozoa and nonprotozoa. It is natural to say that Jocko is neither in the extension nor in the anti-extension of 'animal' and that Jocko is neither in the extension nor in the anti-extension of 'protozoon'; if that is so, then
>
> $$(\text{Jocko is a protozoon} \to \text{Jocko is an animal})$$
>
> will be neither true nor false. Hence,
>
> $$(\forall x)(x \text{ is a protozoon} \to x \text{ is an animal})$$
>
> will be neither true nor false.
>
> Jocko's story is fictional, but it is a realistic fiction ... if we do not have any good reasons to suppose that there is no creature in the position in which we have imagined Jocko, then we do not have any reason to suppose that
>
> $$(\forall x)(x \text{ is a protozoon} \to x \text{ is an animal})$$
>
> is true. The generalisation
>
> All protozoa are animals.
>
> becomes highly suspect.
>
> 'All protozoa are animals' is not an accidental generalisation. It is a basic taxonomic principle that is about as secure as a law of nature could ever be. To forbid the assertion that all protozoa are animals is to outlaw science. (pages 101–102)

This example is spot-on. A three-valued solution of this kind others invalidates things that are particularly fundamental to the way we reason — not only in semantics, or set theory, and other fields where the paradoxes arise, but also in science.

However, there are other paracomplete solutions that have none of these worries. Once we reject the naïve view that the truth value of the conditional is a function of the truth values of its antecedent and its consequent (or we expand the set of truth values to the interval $[0, 1]$, as in Łukasiewicz's infinitely valued logic[6]) we are able to support scientific generalisations even when they include borderline cases. And, we are able to explain the truth of claims such as

All protozoa are animals.

and

If (λ) is true, then (λ) is true.

And even

If a conjunction is true, so are its conjuncts.

which are each taken as truth-valueless on the 3-valued approach. Similarly, a paraconsistent approach need not fall to this objection. So, this objection deals with a naïve approach without a decent conditional, but fails to count against the accounts with more sophisticated logical machinery.

The degradation of methodology. This is the most telling objection. It questions the entire notion of 'changing logic' to give desired results, in this case, a coherent theory of truth (or sets, or whatever). McGee writes:

> [This objection] is based on an admonition of Field [41] that our methodological standards in semantics ought not be any lower than our methodological standards in the empirical sciences. We shall contravene this admonition if we attempt to cover up the deficiencies of our naïve theory of truth by abandoning classical logic.
>
> Imagine that we have a genetic theory to which we are particularly attached, perhaps on political grounds, and that this theory tells us that, if a certain DNA molecule has an even number of nucleotides, then all fruitflies are brown; that, if that particular molecule does not have an even number of nucleotides, then all fruitflies are green; and that fruitflies are not all the same colour. It would surely be absurd to respond to this circumstance by saying that our cherished genetic theory is entirely correct and that classical logic does not apply when we are doing genetics. What we have to say instead is that the genetic theory has been refuted...
>
> As preposterous as it would be to respond to the embarrassment faced by the genetic theory by saying that classical logic no longer applies when we are doing genetics, it would be no less preposterous to respond to the liar paradox by saying that classical logic no longer applies when we are doing semantics. The liar paradox refutes the naïve theory of truth. It is our duty to come up

> with a better theory of truth. It is a dereliction of duty to attempt to obscure the difficulty by dimming the natural light of reason. (pages 102–103)

The first and last of these three objections have some force. To answer them, we need to take an excursion into the philosophy of logic.

1.6 Formal Logic and Reason

Many of the comments about deviant logic and the rationality of a deviant approach to the paradoxes stem from a fundamental misconception of the nature of formal logics, and their relationship to reason and rationality. In this section I shall sketch an account of formal logic which will help us evaluate McGee's criticisms. This account follows the lead of Susan Haack in her *Deviant Logic* [56], who writes:

> ... logic is a theory, a theory on a par, except for its extreme generality, with other, 'scientific' theories ...

And she is right. A system of formal logic, when you think of it, is simply a theory. It is not different in kind from any theory in physics, biology or sociology. What is different is its subject matter. Formal logic is about arguments. In particular, valid arguments. The goal of any formal logic is to provide us with a way of representing arguments in a formal system, and to give us a principled way to distinguish valid argument forms from invalid ones. Haack continues the analogy by taking a pragmatist view of logic, because she is sympathetic to the pragmatist view of the physical sciences. Unlike Haack's, our account of logic is not tied to any particular view of the philosophy of science.

EXCURSUS: Perhaps another lesson could be learned from the conception of logic as science. It seems to follow from this view that the issue of what it is that *makes* arguments valid, the *ground* of logical validity — whether it's just the meanings of the logical connectives, or human convention, or their status in our web of belief — does not have to be answered by a formal logician. This is not to say it isn't an interesting and relevant issue. The analogy can be made with physics. What it is that *makes* the universe the way it is, and the *ground* of physical law is not an issue for physical theories. For example, the general theory of relativity is consonant with the view that laws are Humean regularities, and the view that laws are the patterns in the action of a Deity who sustains the universe, and the view that laws are the result of some other kind of necessity. The physical theories constitute a description of the Way The World Is, without giving a metaphysical description of Why it is that particular way. Similarly, the ground of logical validity, although an interesting issue, is largely independent of the task of logical formalising.

One important exception is the possibility that our logical theorising itself has some effect on the truthmakers of valid argument — which may be the case if some kind of logical conventionalism is true. There are many interesting issues here, which need space of their own in order to do justice to them. □

It should be easy to see that on this view it is wrong to equate classical propositional logic with Reason. Classical logic is a *theory* about the validity of arguments. Similarly, intuitionistic logic, Łukasiewicz's three-valued logic, and any of a whole horde of formal systems are theories about a particular class of valid arguments. It would be as wrong to equate classical logic with Reason as it would be to equate the general theory of relativity with the Way The World Is. The general theory of relativity may describe the Way the World Is in a clear and perspicuous way, it may fit the facts, or be ideally useful, or maximally coherent, or whatever — but it isn't to be identified with what it is intended to describe. Similarly, classical logic ought not be identified with what it is intended to describe, no matter *how* successful it is thought to be, or how widely accepted it is. (As common-sensical as this point is, it is not often acknowledged in the literature. A particularly clear statement of this thesis can be found in Chapter 14 of Priest's *In Contradiction* [113].) This point can be developed further, to deal with Quine's objection that changing the logic is changing the subject. Check Section 1.8 in this chapter to see how the answer can proceed.

In fact, the formalism of classical logic on its own does not even amount to a theory of valid argument. It must be coupled with a principled collection of translation rules, which can provide a reasoned justification for the formalisms that are chosen for each natural language argument. This is a highly non-trivial task. As an example we need to give some non-question begging account of why the rule

From $p \wedge q$ you can validly deduce q.

doesn't licence the deduction of

I'll shoot.

from the premise

One false move and I'll shoot.[7]

I have no doubt that such an explanation can be given. But the fact that the translation rules from natural language into the formal system ought to be provided, and that this is not a trivial task, at least deserves mention.

Now what would it be for a formal system to be correct? It would have to account for our valid argument (in its domain). In other words, for any of our valid arguments in the domain of the logic in question, there should be a formalism that accounts for its validity. Similarly, for any invalid argument in the domain in question, the formalism should be able to deliver some kind of counterexample, so as to explain its invalidity. So much seems clear.

Let's apply this to the case of classical logic, to see how it fares. Minimally, for classical logic to be 'correct,' it should be that anything formalised as a theorem of classical logic turns out as true on interpretation. In general it would be impossible to check this, for there is an infinite number of such statements. However, some kind of recursive procedure might enable us to convince ourselves of this fact. And this is how we impress the truth of classical logic on our first-year logic students. We tell them stories about truth values, and truth value assignments, and we show them the truth-tables of the connectives, which give a recursive procedure for generating the theorems and valid rules of classical logic. This procedure is reasonably convincing (except for the table for "$\supset$," of course). But is the story correct? What's more, is the story so clear that no alternatives are to be countenanced under any circumstances? Clearly not. This kind of introductory presentation of classical logic contains assumptions that can be rationally doubted, and that alternative propositional logics should be countenanced.

Firstly, in one basic presentation of classical logic, an *evaluation* is defined in terms of a mapping from the set of sentences of the formal system into the values T and F. On interpretation this means that each proposition is either true or false — the *Principle of Bivalence*.[8] It is easy to convince Logic I students of this principle. This is often achieved by presenting them with example statements such as

Snow is white.

Queensland has won the Sheffield Shield.[9]

$2 + 2 = 4.$

If, on the other hand, we showed our students statements like

The size of the continuum is $\aleph_1$.

Graham Priest is taller than Bilbo Baggins.[10]

That colour patch is red. (When pointing to a borderline case)

The present King of France is bald.

There will be a sea battle tomorrow.

This electron is in position x with momentum p.

This sentence is false.

The property "heterological" is heterological.

The set of all nonselfmembered sets is nonselfmembered.

we may at least elicit *indecision* about the principle of bivalence. (Some of the best minds have at least been hesitant about bivalence in these cases.) Now it might be thought that each of these examples can be explained under the hypothesis of bivalence — and so we have no reason to reject bivalence — but this would be beside the point. At this stage of our logical theorising, bivalence is not something we can defend, for it is not in our possession. Bivalence is an assumption that needs to be argued for just as much as any alternative to it. Robert Wolf gives a helpful illustration [168].

> It is the lack of positive support for classical logic and, more importantly, the fact that there is no felt need to support classical logic as more than a mathematical system that is the unspoken assumption of most of the discussions of rival logics, including Haack's [55]. It is generally assumed — and very rarely argued for — that classical logic is itself philosophically acceptable and that the rival logics must dislodge classical logic before they are acceptable as more than just curiosities...

> The conceptual situation can perhaps be captured in an image. Defenders of classical logic are like soldiers in a heavily entrenched fortress, while proponents of rival logics are like besieging forces intent on razing the fortress to erect their own on the spot. In the absence of overwhelming force and complete victory, the fortress stands and the defenders remain undislodged. Arguments on rival logics operate on a "possession is nine-tenths of the law" principle, placing the entire burden of proof on those in favour of a rival logic. The proponents of classical logic need only take up a defensive stance and snipe away at the enemy without venturing forth and putting their own positions into question.
>
> It should be apparent from the images chosen that another view is possible. It need not, and we think *should not*, be taken for granted that classical logic is itself any more acceptable than its rivals.

Classical logic is a simple formalism that has difficulty with accounting for all the facts, but became popular since its inception. It is not *prima facie* superior to all other logical systems. In fact, classical logic shows all of the signs of being a degenerating research paradigm. (See Priest's "Classical Logic Aufgehoben" [114] for an interesting argument for this claim.)

The process of formalising logical systems all too often involves feeding our intuitions with simple cases of purported 'laws,' like bivalence, to convince ourselves that they hold in general. *Then* we try to resolve into our scheme cases that don't seem to fit. Sometimes this strategy works, and it is interesting to see how odd statements can be handled in a classical manner, but its just as important to see what can be done without the simplifying assumptions of classical logic. In the presence of odd statements like those we have seen, it is as important (and rational) to consider formal systems that are not founded on the principle of bivalence as those that are. Bivalence is a substantive and significant claim about propositions. If we have formal systems that can model our valid reasoning, yet are properly weaker than classical logic, we have a reason to adopt them over and above classical logic, all other things being equal, because these systems make *fewer* assumptions about propositions. It is an open question as to whether or not deviant logics not founded on the principle of bivalence can model our own valid argument. I will sketch a reply to this soon. Before this we must deal with another kind of deviance from the classical norm.

This deviance centres on matters of inconsistency. I must admit, it is hard to see what it would be for a contradiction to be true. But faced with the liar, and either without a prior training in classical logic or an open mind, someone could be convinced. If this line is taken, some kind of response has to be made to the arguments that from a contradiction you can validly derive anything. Admittedly, this is an odd artefact of the classical apparatus, but there is at least one interesting argument to this conclusion, due to C. I. Lewis and C. H. Langford [68].

$$\dfrac{\dfrac{p \wedge \sim p}{\sim p} \qquad \dfrac{\dfrac{p \wedge \sim p}{p}}{p \vee q}}{q}$$

The most suspicious looking rule in this context is the deduction of q from $\sim p$ and $p \vee q$, famously called *disjunctive syllogism*. How is disjunctive syllogism justified? Most often as follows: $p \vee q$ is true, so it must be either that p is true or q is true. We have $\sim p$, so it can't be p that's true — so it must be q that's true. This seems like a plausible argument. (Let the fact that it's simply another instance of disjunctive syllogism be ignored for the moment.)

How does this justification fare in the context of the Lewis and Langford argument? It simply doesn't apply. Why? Because the reasoning breaks down at the step from the truth of $\sim p$ to it not being p that makes the disjunction true. Under the assumption of the truth of $p \wedge \sim p$, this fails. Under this assumption, it *is* p that grounds the truth of $p \vee q$, so we can't just go ahead and deduce q. Lewis' argument is not going to convince sceptics, who wonder why it would be that a contradiction would entail anything at all, provided that the sceptics are reflective enough to ask why it would be that some take disjunctive syllogism as valid.[11]

As with bivalence, it is interesting to see how much reasoning can go on without assuming consistency. A fair amount of work is going on in this very area, some of it *very* interesting [93, 99, 113, 117, 115, 152].

To sum up this view of the nature of formal logic, I'll use a remark by John Slaney, who argues for the rationality of the enterprise of deviant logic [152].

> The starting point of all logic is the question of which are the valid (perfect, reliable, necessarily rational) forms of argument. What we do in answer to this question is to think up some argument forms which seem good to us, isolate what we take to be the logical constants involved, formulate rules of inference to govern the behaviour of these and thus arrive at a formal calculus ... We somehow have the impression that our logic is inexorable, so that to question it is not even intelligible. But clearly this inexorability is an illusion. The formal theory goes a long way beyond the intuitive reflections that gave rise to it, so that it applies to many arguments of sorts not considered at all when we so readily assented to the rules ... when we considered resolution or the disjunctive syllogism we may have thought: yes, I reason like that; I would regard it as quite irrational not to. But of course, we were not then thinking of reasoning situations that involve taking inconsistent assumptions seriously.

There are enough problems at the core of the project of logical formalisation to cast doubt on the primacy of classical logic. Classical logic is not something that we have fixed and established by a huge weight of evidence, that is beyond dispute — or even something that we need a great deal of evidence to 'dislodge.' It relies on a number of generalisations that might seem initially plausible, but have trouble dealing with all the data at hand — especially the paradoxes of self reference.

The methodology of formal logic is (or ought to be) some kind of inductive procedure involving the gathering of plausible argument forms, the formation of systems that capture these forms and somehow explain why these argument forms are valid, and then the testing of these formalisms against more data. None of this procedure is beyond criticism — especially in the context of such problems as the paradoxes of self reference.

From this perspective of formal logic I'd like to echo a famous plea: let a hundred flowers blossom. It should be obvious that dogmatism is out of place in logic. The rational approach is to consider a menagerie of formal systems, and see how each fare in a wide range of reasoning situations. However, I should make clear that I'm not holding that nothing is fixed or firm in logic. It doesn't seem to be the case that 'anything goes'. Some might think this is the case [100] but it really does seem that a number of rules follow immediately from the way we use the logical connectives. These seem to be at least

$$p \wedge q \Vdash p, \quad p \wedge q \Vdash q,$$
$$\text{If } p \Vdash q \text{ and } p \Vdash r \text{ then } p \Vdash q \wedge r,$$
$$p \Vdash p \vee q, \quad q \Vdash p \vee q,$$
$$\text{If } q \Vdash p \text{ and } r \Vdash p \text{ then } q \vee r \Vdash p,$$
$$p; q \Vdash r \text{ if and only if } p \Vdash q \to r.$$

(Where '$\Vdash$' represents logical consequence, and the semicolon represents some kind of premise combination. See the next chapter for an account of logics like these.) This formalisation gives us a number of core logical principles that seem beyond doubt; they seem to survive, given whatever odd propositions you substitute into them. It is broad enough to encompass the vast majority of systems seriously proposed as propositional logics such as classical logic, paraconsistent logics, relevant logics, modal logics, intuitionistic logic, Łukasiewicz's logics, linear logic and quantum logic. My thesis is that we just *don't* have enough information to decide between each of the systems in this range. We ought to compare and contrast each of the systems in this family, and see their strengths and weaknesses as models of our own valid argument. Before using this picture of formal logic to deal with McGee's arguments, let me fend off a few objections to this picture.

Aren't Mathematicians Classical Logicians? It may be objected that the only way to make sense of 20th Century mathematical practice is to assume the validity of classical logic. After all, if mathematicians prove truths using valid means, and they avail themselves of all the moves of classical logic, then we ought to take these moves as valid.

This is an interesting argument, but it doesn't deliver its conclusion. Mathematical reasoning is interesting in a number of respects. Firstly, by and large mathematicians treat their subject matter as consistent and complete. It seems that classical mathematicians have an assumption that for every proposition p that they consider, either p is true or $\sim p$ is true. To make sense of this practice we have no need to take $p \vee {\sim} p$ as a theorem, we simply can take it as an assumption that mathematicians make, and show that they validly reason from there. In addition, mathematicians seem to take it that if p and $\sim p$ were true, that would be disastrous for their subject matter. So, another of their assumptions is $p \wedge \sim p \rightarrow q$ for every q. To make sense of mathematical practice, we need not take these claims to be theorems of our logic — we need just add them as assumptions, and then note that under these assumptions their reasoning can be seen as valid.

This is not quite all. The conditional that is used in mathematical contexts is notoriously non-modal and irrelevant. Moves such as deducing $q \rightarrow p$ from p are widespread in mathematical contexts. To explain this we must either equip our logic with a conditional that will validate the required moves, or simply assume them for mathematical propositions. Again, this is not a problem. Some may balk at this proposal, but it merely represents mathematical reasoning as enthymematic. (A conditional 'if p then q is said to be *enthymematic* if there are some extra true statement r such that q follows from p and r. The true r (which can be quite complex) is said to be the *enthymeme*.) The enthymemes are unproblematic for the mathematician, who will readily assent to them if asked. They are all theorems of classical logic. This places mathematics on as strong a footing as does the classical position, and in this way we can make sense of mathematical practice. If there is something suspect about the propositions that are taken as the enthymemes, this is just as much a problem for the classical account of mathematical reasoning as it is for this one. For all of these enthymemes come out as particularly simple classical tautologies, which are taken to be beyond doubt and obvious by the classical orthodoxy. There's no difficulty with a deviant saying that the classical account is right as far as these *instances* of classical laws are concerned. Their illicit generalisations are the problem.

So, there is no need for the deviant to engage in a *revisionist* programme in mathematics. Mathematicians' reasoning can be explained from the perspective of deviant logic, without having to conclude that mathematicians have 'got it wrong' at any stage. This option is open — a deviant may point to an assumption that has been made in some mathematical context and ask whether or not it is warranted, as constructivists do — but it is not forced by the acceptance of a deviant logic.

EXCURSUS: Mathematical reasoning need not make consistency or completeness assumptions. There is a strong history of constructive mathematics, that seeks to reconstruct mathematical reasoning without making nonconstructive assumptions (including bivalence). There is also a more recent history of work on *inconsistent* mathematics, wherein inconsistent but non-trivial mathematical theories are tolerated. Some of the results in this field are noteworthy. As an example, there is a finitary non-triviality proof of Peano arithmetic in a strong paraconsistent logic — a result which is notoriously impossible in classical arithmetic [89, 93, 99]. □

Isn't Excluded Middle Plausible? A similar objection can be given which is closer to home territory for most of us. How do I explain the intuitive pull that the law of the excluded middle has, if it isn't a logical truth? For example, I believe that either I've read all of the *Nichomachean Ethics* or not — without believing either disjunct (I just can't remember). Am I warranted in this belief?

Clearly I am. What we want is an explanation as to why. From the perspective of deviant logic, there are two. Firstly, it doesn't follow that because the law is not a theorem of my favoured logical systems, I cannot rationally believe many of its instances. It seems that for the vast majority of events in my vicinity, either they happen or they don't. I'm quite rational in believing that the same is true in this case. In fact, the prevailing truth of excluded middles in the general vicinity of my world of medium-sized dry goods (where I don't look too closely at the borderlines of vague predicates) might lead me to think that they are generally true in that area of the Way Things Are. But the further away from my world I go, into the upper reaches of set-theory, or to meditations on the properties of truth in semantically closed languages, or the problematic borderlines of vague predicates, my expectation of the truth of the 'law' fades. None of this is particularly irrational — in fact, it's more cautious than the classical approach.

The other explanation which may pay off in a different way is to note the problems of translating into 'formalese.' My utterance of "p or q" may be better formalised as $\sim p \rightarrow q$ instead of $p \vee q$. After all, when we utter a disjunction, it often has the force of "if it isn't the first disjunct that's true, it's the second." In this case, my utterance of "p or not p" could be formalised as $\sim p \rightarrow \sim p$, which is a logical truth. Of course, in most deviant logics, this formalisation of disjunction is not going to satisfy everything that garden-variety extensional disjunction does, but it may be more appropriate for formalising some of our utterances. (This distinction goes back to Anderson and Belnap [1].)

Aren't there Arguments for Excluded Middle? The deviant logician may have another problem. Perhaps the choice of a formal logic should not proceed by way of inductive generalisation. Maybe there are good *a priori* arguments for particular logical laws. This might be thought to be the case with the law of the excluded middle. Perhaps some deep thought about the nature of the bearers of truth might give us a valid argument whose conclusion is that every proposition is either true or false. I do not deny that this is possible, and any 'deviant' response to such arguments must be on a case-by-case basis. However, some programmatic remarks can be made as to how these cases are dealt with.

The main weakness in such arguments is that they seem to invariably beg the question. Arguments for the law of the excluded middle without question seem to rely on instances of that law to get to the conclusion. This isn't begging the question or illegitimate as such, for the particular instances may be less problematic than the generalisation. This is where the arguments may have some bite. However, it seems to be the case that the particular instances that are used in these arguments are just as problematic as what they attempt to prove, and it is *this* that is begging the question.

1.7 Answering McGee's Objections

Let's use what we have seen so far to answer McGee's objections.

The difficulty of learning the 3-valued logic. The first answer to make to the objection is that by-and-large we do not reason by using a formal system. This is to put the cart before the horse. The formal system is there to explain and model our reasoning, to give us an insight into the reasoning we already do, and to perhaps aid us in it. The deviant logician attempts to model the *same* reasoning as the classical logician. To engage in Reason, we do not have to learn *any* formal system, whether classical or deviant. However, the objection can't be brushed aside immediately. McGee reiterates an objection made by Feferman [40], that in the 3-valued logic under consideration:

> Nothing like sustained ordinary reasoning can be carried on. (page 100)

Now even if we note that nothing like sustained reasoning really happens in first-order predicate logic either (the vocabulary is very poor), McGee still has an objection. It can be rephrased as the claim that deviant formal systems cannot account for the sustained ordinary reasoning we regularly engage in, in contrast to classical logic, which can. As we've already seen while dealing with the objections from the practice of mathematics, this is not true. Many deviant logics can formalise our ordinary reasoning without difficulty, given a few plausible assumptions that we would probably agree with anyway. Provided enough instances of the law of the excluded middle are assumed, the 3-valued logic that Feferman and McGee object to becomes as strong as classical logic. And so, reasoning can be explained from that point of view, if it can be explained classically. It only differs in that the arguments used are interpreted as enthymemes. So, this objection does not have any force against the deviant position.

The degradation of methodology. Recall McGee's objection to the practices of the geneticist who rejects a logical law in order to keep alive a favoured theory. It is simple to produce a similar example with a very different point.

> Imagine we have an entomological theory to which we are particularly attached, perhaps on political grounds, and that this theory tells us that, if one fruitfly sets off in a straight line to a mango tree, and sends out a particular signal to a fruitfly some metres away (not on its flightpath) then this second fruitfly will make a *parallel* journey in the same direction, but that the theory also tells us that the two fruitflies will meet at their destination. It would surely be absurd to respond to this circumstance by saying that our cherished entomological theory is correct and that Euclidean geometry doesn't apply when doing entomology.

This is just as convincing as McGee's example. It is as silly for an entomologist to deny claims of geometry that are quite acceptable in their domain for the sake of a cherished theory as it is for a geneticist to deny claims of a logical nature that are quite acceptable in that domain. Yet just this century it was rationally countenanced that Euclidean geometry doesn't apply when doing cosmology. So the argument form that McGee uses doesn't deliver his conclusion. A case has to be made as to what makes logic differ from geometry in some relevant respect. No case like this has yet been given. So as things stand, if this argument works, it works as much against the practices of modern cosmology as it does against those of us using deviant logics in our analyses of the paradoxes.

Excursus: The comparison between geometry and logic is a fruitful one, which can tell us a great deal. In geometry, some enterprising mathematicians considered systems which differed with respect to the parallel postulate. Years later, these geometries proved useful to theories in physics. This shows that our theories about points and lines can vary quite a lot and make sense *as geometries*, and not merely as formal abstractions. The case of non-classical logics is similar. Our theories about conjunction, disjunction, negation and implication can vary in just the same way. Formal systems satisfying the minimal conditions on connectives that we saw before all seem to make sense *as logics*, and not as merely formal systems.

The analogy also reinforces the intuition that formal systems cannot vary endlessly and still remain logics. Just as a system in which points did not lie on lines would fail to count as a *geometry* of points and lines, a system in which $p \wedge q \Vdash p$ and $p \wedge q \Vdash q$ failed would not count as a logic of *conjunction* (at least concerning $\wedge$). □

McGee's objection is a little stronger than what we've seen so far. He countenances the case where the sheer weight of scientific observation might convince us to abandon classical logic. What makes semantics less successful than the other sciences is that the data is so scarce that it cannot apply the needed pressure. He writes:

> In genetics we have a huge body of empirical data that our theories are attempting to explain. We can imagine this body of data by its sheer bulk pushing classical logic aside ... Now, the pressure to abandon classical logic in semantics does not come from an overwhelming body of linguistic data but rather from our metaphysical intuitions about truth. In metaphysics, we scarcely have any data. All that we have to take us beyond our preanalytic prejudices is our reason, and now we are asked to modify the rules of reason so that they no longer contravene our preanalytic prejudices. In the end, the role of reason in metaphysics will be merely to confirm whatever we have believed all along. (page 103)

There's some nice imagery there, but it won't do the job. Classical logic is not the heavy bulk of Reason that has to be pushed aside. It is merely a theory about Reason. We have seen that classical logic is grounded in metaphysical intuitions about truth, just as semantics is. Reason itself is never to be moved about, but it is not clear what Reason itself has to say in the case of the liar paradox. We can argue from the liar and the T-scheme to the truth and falsity of the liar from the premise that it is either true or false. Reason at least helps us deduce that. Some say that Reason assures us that the liar is either true or false, and that there is No Way that it could be both true and false. If that is the case, then assuredly, the liar gives us a refutation of the T-scheme. However, as I have argued, it is not obvious that this is what Reason has to say. To take a deviant approach to the paradoxes is not to abandon Reason, but only to question a particular formalisation of Reason.

1.8 Answering Other Objections

Dealing with McGee's objections, and positing a different picture of logical theorising is not enough to divert all criticism of the project of non-classical logics. There are two other objections that are often put against the project, and we will do well to apply the picture of logic we have seen to indicate how these objections fail.

Changing the Logic, Changing the Subject Quine has argued that there really can be no disagreement in logic, because when two parties attempt to disagree on logical principles, they end up talking about different things, and so, they do not disagree. Quine writes, concerning a discussion between a paraconsistent logician and a critic.

> My view of this dialogue is that neither party knows what he is talking about. They think they are talking about negation, '$\sim$', 'not'; but surely the notation ceased to be recognizable as negation when they took to regarding some conjunctions of the form '$p \wedge \sim p$' as true, and stopped regarding such sentences as implying all others. Here, evidently, is the deviant logician's predicament: when he tries to deny the doctrine he only changes the subject. [120]

Let's consider what Quine must mean for the argument to succeed. Quine must be saying that the meanings of the terms involved are constituted (at least in part) by their use, and that here, the uses are so distinct that the terms under discussion (especially different forms of negation) are no longer the same. The parties to the discussion are talking about different things. The argument could be clarified and fleshed out in a number of ways, but these need not concern us. However the argument is developed, it essentially involves a confusion of logic as theory of validity, and logic as the collection of validities we theorise about. If this distinction is maintained, then it is clear that it is possible to differ at the level of theory while agreeing that the competing theories are about the same thing.

An example is in order. Suppose that we have two scientific theories of planetary motion. One, a simple Newtonian theory, in terms of gravitational acceleration obeying the inverse square law, the other, an odd theory that posits gravitation as a force which has markedly varying effects throughout the universe. According to this theory, gravitation obeys the inverse square law of attraction here, but further out in space, it obeys an inverse cubed law, and far away in other galaxies, it is a force of *repulsion*. This is a particularly odd theory, with nothing obvious going for it (except for the possibility of being the background physics for some really interesting science fiction stories). Pretty clearly, this odd theory and the Newtonian theory disagree about the nature of gravitation. The theories are in conflict. However, if we take Quine's criterion for identifying the subject matter of a discussion seriously, we may be forced to conclude that these theories do not disagree because they do not have the same subject matter. Quine could say 'the subject matter ceased to be recognisable as gravitation when they took to regarding it as possibly a force of repulsion and not attraction.' Clearly this reply is out of place. The theorists are talking about the force responsible for planetary motion, and they are positing different properties of it, 'in the large'. The two theories disagree, because they are theories *about* some agreed subject matter. If the odd theory is intended to give us an account of what gravitation is, and it is used in this way, then it disagrees with the standard, Newtonian theory.

The same holds for logical theories. Before any logical analysing was done, there was practice of deduction and argumentation. The question was asked: how can we give an account of the valid arguments? Formalisation and theorisation followed. Eventually, different formalisations of our logical practice arose. These *conflict* if they offer different analyses of the validity of our everyday argument. For example, given a particular instance of arguing from a contradiction to an unrelated proposition, the paraconsistent logician can disagree with the classical logician about the validity of *that* argument. For example, take the liar sentence (λ) from before. One argument that the paraconsistent and classical logicians will disagree about is the following:

> (λ) is true and (λ) is not true, therefore Queensland has won the Sheffield Shield.

The difference between the two logicians is not one of formalisation alone, but one of differing outcomes when using formalisation to model actual arguments, like this one.

Simplicity Another argument for the primacy of classical logic sometimes put forward is the argument from simplicity. It goes (crudely) like this: A simpler logic is a better logic, classical logic is the simplest logic there is, so classical logic is the best logic there is.

There are a number of responses to this argument. Firstly, let's grant that at least as a methodological point, simplicity is desirable. It would be strange to favour a theory which introduced complications if there were no payoff as a result of these complications. Granting this point, the argument is unsound, for a number of reasons. One is that classical logic is not the simplest logic. It is often said that classical logic is simple in that it has a clear semantics, it is decidable, and has various other desirable formal properties. This is true, but there are other, simpler logical systems. Meyer has drawn our attention to the *pure calculus of irrelevance* [84]. It is a simple system, in which every statement is a theorem. This system has all of the nice properties of classical logic. It has a simple semantics (assign everything the one designated value), it is decidable (trivially), it is compact, and so on. Of course, the pure calculus of irrelevance is not a better logic than classical logic, because it does badly in terms of faithfulness to the data.

But then, the defender of non-classical logics can also argue that her favourite logic does even better with the facts than classical logic can. If this is the case, then simplicity can only be a guide, and not a rule, lest the classical logician be faced with the burden of endorsing the pure calculus of irrelevance.

In addition, simplicity of logic is not enough by itself. A logic like classical first order logic does not stand on its own. It must be a part of a larger semantic theory, giving an account of truth, sets, numbers, and our discourse on any of a range of topics. We have already argued that the accounts of the paradoxes retaining classical logic are neither successful, nor simple. In this broader field of study, the measure of simplicity may actually swing in favour of the deviant logician.

Of course, methodological canons like concern for simplicity, harmony, elegance and the like have their place. In later chapters, I wish to show that logics I favour (those without contraction) are simple and elegant by showing how their formal structure arises in a number of different disciplines. They are naturally occurring formal systems which can be used to model a wide variety of phenomena. This is at least helpful information, indicating that study of these systems is worthwhile and fruitful.

1.9 Conclusion

By now it should be clear that deviant accounts of the paradoxes are coherent, and as well grounded in terms of methodology as any approach that takes classical logic as somehow 'privileged.' I hope to have also made clear that any claims of such privileges for classical logic are unfounded. Classical logic is a formalism that has served well in limited domains (principally, classical mathematical reasoning) but which is founded on general principles that are doubtful at best. At worst they are ill-founded generalisations which are in need of replacement.

Once this is granted, we are not committed to being irrational, or to reject truths which we have long held dear. Instead, it brings us to treat the project of logical formalisation in the same way as we do any other science. The task is to construct theories, and test them against the data we have. In this, the paradoxes are most useful, as Russell has taught us. Instead of taking classical logic as a *given*, to which any account of paradoxes must conform, we would do well to take the paradoxes as what they are — experimental data to deal with as a part of the task of providing an adequate account of valid inference. Given the baroque structures that emerge when the paradoxes are treated in the context of classical logic, we can be sure that such an adequate account is not classical.

1.10 Notes

Material in this chapter was presented to the Department of Philosophy of the University of Queensland, and the 1992 Australasian Association of Philosophy Conference [123]. I'm grateful for comments and criticism from those present; especially Graham Priest, Gary Malinas, Ian Hinckfuss, Mark Lance and Lloyd Reinhardt. The paper has a second half "Comparing Deviant Logics" that was read at the 1992 Australasian Association for Logic Conference, held in honour of Bob Meyer. Both are dedicated to him with appreciation for his work. The inspiration for this approach is varied. It has arisen out of reading the work of Quine and Haack, enticed by their ideas, but fundamentally, not convinced by them. It has been honed and developed in many discussions with Graham Priest, whose views mine are closer to than any others.

[1]Also called the paradox of the heap. We shall examine this paradox a little more in Chapter 8, and see that it also motivates the rejection of contraction.

[2]If you are one of those who do not like this use of the word "sentences" — which includes me at times — feel free to replace the word with a suitable substitute. I find "statement," or "proposition expressed by this sentence in this context" to work marvellously. However, in all honesty, the answer you give to the question of the nature of the bearers of truth, or the contents of propositional attitudes are tangential to most of the issues in this chapter. [Note added in July 1994. J. M. Dunn has convinced me that this way of describing the point is wrong. "Everything Bob says is true" entails that Bob was correct, whereas the conjunction of everything he said, need not. However, the co-entailment of p and $T\langle p\rangle$ remains.]

[3]Subject to the obvious qualifications to the effect that T is relative to the language in which the sentences are expressed. Gupta [54] and Peacocke [107] deal with these subtleties.

[4]Kripke, to be sure, would not like his proposal to be characterised as espousing the use of a deviant logic. In his "Outline of a Theory of Truth" [63] he expresses surprise that some people have so described his position. His defence is that sentences expressing propositions behave in a purely classical way. Only the odd sentences that fail to express propositions receive the value "neither true nor false". This is some kind of defence, but as the odd sentences are meaningful, can be believed (in some sense), and can function in valid arguments, this defence is not convincing. The logic for determining the validity of arguments involving these sentences is not classical, and so, the proposal is deviant.

[5]Subsequent references to *Truth, Vagueness and Paradox* will be by page number only.

[6]See Chapter 3 for a definition of Łukasiewicz's infinitely valued logic.

[7]I owe this example to John Slaney.

[8][Added in July 1994: J. M. Dunn noted that this point was also made by Arthur Pap, years ago [174].]

[9]Well, only in Australia, and only when looking for an obvious falsehood.

[10]The difficulty is that Graham Priest is an existing (rather tall) human being, and Bilbo Baggins is a fictional hobbit.

[11]This analysis of disjunctive syllogism and Lewis' argument is given by J. Michael Dunn [31] and John Slaney [152].

Chapter 2
Basic Proof Theory

Proof.
Some people gonna call you up
tell you something that you already know.
— PAUL SIMON 'Proof' [142]

Let's collect together some threads from the introduction and the first chapter. From the introduction we know that logics without contraction are interesting. From the previous chapter we know that it is worth studying a whole range of formal systems. In this chapter we will define the systems that will be the focus of our attention in the rest of this work. These range from being quite weak (validating very few inferences) to being quite strong (validating rather a lot), but they have one thing in common. They all reject contraction.

2.1 Notation

Whenever you define a formal system for modelling deduction, you need a *language* for the system. We will define the language $\mathcal{L}$ of our systems in the usual way. (If tedious recursive definitions bore you, skip to the asterisk in the margin on the next page, where things are more interesting.) Our systems are designed to give an account of the valid argument expressed in the connectives

$$\wedge \qquad \vee \qquad \rightarrow \qquad \sim$$

(conjunction, disjunction, the conditional (or entailment) and negation respectively — each are binary connectives, apart from negation which is unary) and the universal and existential quantifiers

$$\forall \qquad \exists$$

Those are standard. It will be very handy to have around two extra connectives, which are not so standard.

$$t \qquad \circ$$

The first is 0-ary (a sentence constant denoting a particular truth) and the second is binary. This is the fusion connective, of which we shall see much more, a little later.[1]

We'll take our language $\mathcal{L}$ to feature a countable supply of variables $\nu_0, \nu_1, \nu_2, \ldots$ which will range over the domain of quantification. To state things about the objects in our domain of quantification we use the predicates F_ξ (for $\xi < \alpha_1$, where α_1 is some cardinal), such that F_ξ has arity $\mu_1(\xi)$. So, $\mu_1 : \alpha_1 \rightarrow \omega$ is the *predicate arity function*. Sentence constants are predicates of arity zero.

Our language also features a family of function symbols f_ξ for $\xi < \alpha_2$ (α_2 another cardinal), such that the f_ξ is of arity $\mu_2(\xi)$. And $\mu_2 : \alpha_2 \rightarrow \omega$ is the *function arity function*. Constants are functions of arity zero.

The language, and the set of terms in the language is then recursively defined in the usual way. First by defining terms, and then by defining formulae. A term is either a free variable, or a function applied to the appropriate number of terms. An atomic formula is a predicate applied to the appropriate number of terms, and then a formula is one formed from atomic formulae by the usual recursive procedure using connectives and quantifiers. We allow vacuous quantification and free variables in formulae, so that $\forall v_0(F_0(v_1, v_2))$ counts as a formula (provided that F_0 has arity 2).

We eliminate a few parentheses by the usual conventions. Negation binds more tightly than fusion and the extensional connectives ($\wedge$ and $\vee$) which bind more tightly than the conditional. So, $\sim p \wedge q \to r \circ s$ abbreviates $((\sim p) \wedge q) \to (r \circ s)$.

For ease of reading, we'll take a, b, c, ... to be shorthand for the first few 0-ary function symbols (or constants), p, q, r, ... shorthand for the first 0-ary predicate symbols (or sentence constants), and F, G and H, predicate symbols of higher arity. We take x, y, z to be metavariables ranging over variables, and A, B, C, ... to be metavariables ranging over arbitrary formulae. So, $\forall x(A \vee B)$ ranges over all universally quantified disjunctions.

We follow the usual story concerning free and bound variables. The scope of an instance of a quantifier Qx in QxA is that instance of the formula A. Similarly, if QxA is a subformula of B, the scope of that instance of Qx is that instance of A. A free variable in a formula is a variable that is not in the scope of one of its own quantifiers in that formula. All other variables in that formula are bound. A sentence is a formula that contains no free variables. Given a formula $A(x)$ with a number of instances of the free variable x indicated, the formula $A(t)$ is given by replacing those instances of x by t. The term t *is free for the variable* x *in* $A(x)$ just when every instance of a variable in t is free in $A(t)$ wherever x occurs free in $A(x)$.

A word is in order about what we have just done. We have defined a formal lan- *
guage that we shall use to give an account of valid argument. As we have recognised in Chapter 1, doing this has simplified things greatly. Natural languages are not so neat or univocal. Not every use of 'and' bears the same meaning or enters into the same kind of inferential patterns as our $\wedge$. We have produced a simplified *model* of a fragment of a natural language. As such formulae are not parts of our natural language. They are *models* of parts of our language. Individual formulae are not sentences in their own right. They are abstractions that we use in order to give an account of deduction in our language. To reflect this, we will not use formulae as sentences in their own right. We will not say "if $p \to r$ and $q \to r$ then $p \vee q \to r$." On our account this makes no sense, because $p \to r$, $q \to r$ and $p \vee q \to r$ are not propositions or statements on their own; they are merely models for propositions, or formalisms that are intended to represent statements. Instead, we will use other means to talk about deduction between formulae. We will write

$$(p \to r) \wedge (q \to r) \Vdash p \vee q \to r$$

to claim that in a particular formal system the deduction from $(p \to r) \wedge (q \to r)$ to $p \vee q \to r$ is valid (in a sense to be elaborated later). We use $\Vdash$ as a sentence-forming operator. To put it in another way, in our natural language (English, here) formulae are nouns and '$\Vdash$' is a verb.

EXCURSUS: That said, we will fudge things just a little, to use '$X \Vdash A$' for both the claim that the deduction of A from X is valid (in a system) *and* to name the deduction from X to A itself. The context will make the intended interpretation clear. □

Now we will use the language to express a range of formal systems that give different accounts of valid inference in our language.

2.2 Natural Deduction Systems

Lemmon-style natural deduction systems are one pleasing and intuitive way to formalise valid deduction [66]. These are systems which formalise deduction using a number of primitive rules that are quite like rules we use in ordinary reasoning; *modus ponens*, conditional proof, conjunction introduction, and so on. At each stage of the proof, a record is kept of the assumptions behind the asserted claim. We will give natural deduction formulations of a range of formal systems in this rather long definition.

Definition 2.1 As in Lemmon's original system, the calculi manipulate *sequents* of the form

$$X \Vdash A$$

where X is some structured collection of assumptions, and A is a formula. This means that on the assumption of X, then A follows as a matter of logic. However, unlike Lemmon's original system, we take it that assumptions can be bunched together in two different ways. One mirrors the deduction of $A \wedge B$ from A and B. If a piece of information X warrants A, and Y warrants B, then the result of taking X and Y together warrants $A \wedge B$. So much is clear. This means of 'taking together' is analogous to extensional conjunction.

A notion of 'taking together' also arises from conditionals. Consider the deduction of B from A and $A \to B$. If X warrants A and Y warrants $A \to B$, then taking X and Y together warrants B. However, this kind of taking together is different. In this case, the 'taking together' may well be asymmetric in the sense that it is clear that in the deduction we are *using* Y as an 'inference ticket' providing a deduction from X. It is not so clear that this is what is going on in the first case in which we simple-mindedly merge the information in X and Y. So, let's take there to be two kinds of bunching. The first type we will call *extensional*, and the second, *intensional*. Take X, Y to be the extensional bunching of X and Y, and $X; Y$ to be their intensional bunching.

An example from modal logic might help motivate the difference between the bunching operations. For the moment, let $\to$ be a conditional with some kind of modal force. Then we would not expect that a proposition B would be ground enough to warrant $A \to B$, because assuming the antecedent A may take away the grounds for concluding B. If this is the case, then the intensional bunching operator (associated with this conditional) must differ from straightforward extensional bunching. For otherwise, we would have $A \to B$ following logically from B. We could proceed as follows: Suppose that $X \Vdash B$. Then $A, X \Vdash B$ too, by properties of extensional bunching. (Adding extra information extensionally does nothing to take information away.) If this were the same as intensional bunching, we would have $A; X \Vdash B$ too, and hence, $X \Vdash A \to B$, by the connection between intensional bunching and the conditional. If a conditional has some kind of modal force, then the intensional bunching operator will diverge from extensional combination, in that it will sometimes 'weaken' information. Sometimes $X; Y$ will not warrant all the information that Y would warrant. Intensional and extensional combination must come apart if we are to model a wide range of conditionals.

This approach is important, because it 'cordons off' the behaviour of conditionals from that of the extensional connectives of conjunction and disjunction. We pick out the standard sense of extensional conjunction and disjunction with our extensional bunching operator ',' irrespective of the behaviour of the intensional operators, which are the interesting objects of study.

The approach of using two bunching connectives to model intensional and extensional connectives is by no means new. The idea is due to Dunn [4, 27, 28, 32] and independently, Minc [171], and it was later adapted by Meyer [85] and others (see references in Chapter 7) to apply to a wider range of logics. We will follow the presentation of Slaney's paper "A General Logic" [151], which contains more discussion of general issues than is possible here.

The simple minded merging of two pieces of information is idempotent, associative and commutative. It also 'weakens' in that if $X \Vdash A$ then $X, Y \Vdash A$ too. Adding the information Y doesn't take away any of the information in X. The properties of extensional bunching are easy to pin down. The properties of intensional bunching are not as clear. Does applying Y to X result in the same things as applying X to Y? Perhaps, perhaps not. It depends on the notion of application. Does applying Z to the result of applying Y to X amount to the same thing as applying the result of applying Z to Y, to X? It's not obvious that it does, and it's not obvious that it doesn't. Does the result of applying Y to X contain X, or is some of X's original content 'filtered out'? There are a whole range of possibilities for the properties of intensional bunching, each of which will result in different properties of the conditional. As I argued in the previous chapter, it is sensible to leave these questions open and to deal with a whole range of formal systems.

A proof using this formal system is a list of sequents of the form $X \Vdash A$. Each line in the proof will follow from earlier lines by means of a range of rules that show how sequents follow from other sequents. For example, we know that $X, Y \Vdash A \wedge B$ follows from $X \Vdash A$ and $Y \Vdash B$ by our earlier ruminations about extensional bunching and extensional conjunction.

Our proofs will be made human-readable by using line numbers to represent assumptions, and annotations to indicate the kind of rule used at each stage — an example will be given below. We use parentheses to disambiguate nested bunches (so $(X, Y); Z$ is the intensional bunch of (the extensional bunch of X and Y) and Z), and we write $X(Y)$ to represent a bunch in which Y occurs in a particular place as a sub-bunch. The clause $X \Longleftarrow Y$ means that a bunch of the form X may be replaced anywhere in the left side of a sequent by the corresponding bunch Y. The structural rules governing extensional bunching are the obvious ones.

$$\begin{array}{llcl} \text{eB} & X,(Y,Z) & \Longleftarrow & (X,Y),Z \\ \text{eC} & Y,X & \Longleftarrow & X,Y \\ \text{eW} & X,X & \Longleftarrow & X \\ \text{eK} & X & \Longleftarrow & X,Y \end{array}$$

Intensional bunching is governed by one rule, giving the behaviour of the 'empty' bunch 0, which represents 'logic alone.'

$$\text{0-right} \quad X;0 \Longleftrightarrow X$$

This means that if $X \Vdash A$, then *logic* applied to X also warrants A. This is clear if we take it that the 'warranting' indicates logical deduction. Conversely, if *logic* applied to X warrants A, then X warrants A on its own. So, this axiom is sound. It also gives us a way to represent theorems — things warranted by logic alone. These are the formulae A such that $0 \Vdash A$.

The rule of assumptions dictates that on any line of a proof, a proposition may be introduced, with that very proposition as its only assumption.

$$\frac{}{A \Vdash A}\ \text{A}$$

The rest of the rules govern connectives. Conjunction is the object language correlate of the extensional bunching operator, so we have conjunction elimination and introduction.

$$\frac{X \Vdash A \wedge B}{X \Vdash A} \qquad \frac{X \Vdash A \wedge B}{X \Vdash B}\ \wedge\text{E} \qquad \frac{X \Vdash A \quad Y \Vdash B}{X,Y \Vdash A \wedge B}\ \wedge\text{I}$$

The natural rules for the introduction and elimination of disjunction are similar.

$$\frac{X \Vdash A}{X \Vdash A \vee B} \qquad \frac{X \Vdash B}{X \Vdash A \vee B}\ \vee\text{I} \qquad \frac{X \Vdash A \vee B \quad Y(A) \Vdash C \quad Y(B) \Vdash C}{Y(X) \Vdash C}\ \vee\text{E}$$

The conditional goes with intensional bunching, according to our ruminations. This gives us *modus ponens* and *conditional proof*.

$$\frac{X \Vdash A \quad Y \Vdash A \to B}{X;Y \Vdash B}\ \text{MP} \qquad \frac{A;X \Vdash B}{X \Vdash A \to B}\ \text{CP}$$

Note in these deductions the way that the order of the bunching reflects the directionality of the arrow in the conditional. If X supports A and Y supports $A \to B$, then applying Y to X supports B. This application is written to the right to graphically indicate that the information given as A (by way of X) 'passes through' the information given as $A \to B$ (by way of Y) to warrant B. Conditional proof also respects this directionality. Note that this is *not* the way that it is done in most of the literature. The change to this form is warranted by its clarity, by connections to relation algebras (Chapter 3), the Lambek calculus (Chapter 10) and some of Dunn's recent work on the semantics of relevant logics and its connections with relation algebra [35].

From conjunction and the conditional we form the biconditional (written '$\leftrightarrow$'), where we take $A \leftrightarrow B$ to be shorthand for $(A \rightarrow B) \wedge (B \rightarrow A)$ in the usual fashion.

Fusion uses the semicolon. It is intended to be an object-language correlate of intensional bunching. As such, it satisfies the following introduction and elimination rules.

$$\frac{X \Vdash A \quad Y \Vdash B}{X; Y \Vdash A \circ B}\ \circ\mathrm{I} \qquad \frac{X \Vdash A \circ B \quad Y(A; B) \Vdash C}{Y(X) \Vdash C}\ \circ\mathrm{E}$$

Although it's not obvious how to read fusion in a natural language, it is a very useful connective to have at hand. The most we can say about it is that it is some kind variant of 'and,' with intensional force.

Although this is by no means immune from dispute, in the logics we consider, *modus tollens* and *double negation* characterise negation:

$$\frac{A; X \Vdash B \quad Y \Vdash {\sim}B}{Y; X \Vdash {\sim}A}\ \mathrm{MT} \qquad \frac{X \Vdash A}{X \Vdash {\sim}{\sim}A}\ \mathrm{DNI} \qquad \frac{X \Vdash {\sim}{\sim}A}{X \Vdash A}\ \mathrm{DNE}$$

These rules give strong negation properties like contraposition, and the de Morgan identities. See the *modus tollens* rule MT, which basically dictates that the conditional not only preserves truth forward (as you would expect a conditional to do) but also preserves falsity backward. This is because the left premise $A; X \Vdash B$ is equivalent to $X \Vdash A \rightarrow B$ by MP and CP. Hence, MT dictates that this fact, together with $Y \Vdash {\sim}B$ delivers $Y; X \Vdash {\sim}A$, which means that the conditional $A \rightarrow B$ (warranted by X) does the same work as the conditional ${\sim}B \rightarrow {\sim}A$. We have contraposition in a strong form. These rules are historically important for relevant logics, but we shall see that some of the semantic structures important to the study of these logics do not support this kind of negation. Much more work must be done on different ways of modelling negation, and their different properties. For the duration of this thesis however, we will use this kind of negation, and simply sketch alternatives as we come to them.

The constant t is the language equivalent of the bunch 0. So the rules for t reflect this fact.

$$\frac{}{0 \Vdash t}\ \mathrm{tI} \qquad \frac{X(t) \Vdash A}{X(0) \Vdash A}\ \mathrm{tE}$$

As with fusion, a reading for t is not immediately obvious. The closest that comes to mind is the conjunction of all logical truths. (In classical logics this is indistinguishable from any individual logical truth — but in logics like ours, which pay more attention to the intensional, it isn't.)

Sometimes it is interesting to add an 'absurd' propositional constant $\perp$ (that than which nothing falser can be conceived). Its only rule states that if X warrants $\perp$ then X

warrants *anything*.

$$\frac{X \Vdash \bot}{X \Vdash A} \;\bot\mathrm{E}$$

Finally, quantification is trivial, in that it relies on neither bunching operation.

$$\frac{X \Vdash A}{X \Vdash \forall x A} \text{ where } x \text{ is not free in any formula in } X \qquad \forall\mathrm{I}$$

$$\frac{X \Vdash \forall x A(x)}{X \Vdash A(t)} \text{ where } t \text{ is free for } x \text{ in } A(x) \qquad \forall\mathrm{E}$$

$$\frac{X \Vdash A(t)}{X \Vdash \exists x A(x)} \text{ where } t \text{ is free for } x \text{ in } A(x) \qquad \exists\mathrm{I}$$

$$\frac{X \Vdash \exists x A(x) \quad Y(A(x)) \Vdash B}{Y(X) \Vdash B} \text{ where } x \text{ is not free in } Y(B) \qquad \exists\mathrm{E}$$

These rules give the basic logic **DW**. A proof that $X \Vdash A$ is a proof with A on a line, with X as the only assumptions. Here is a proof that $(A \to C) \wedge (B \to C) \Vdash A \vee B \to C$

1	(1)	$(A \to C) \wedge (B \to C)$	A
2	(2)	$A \vee B$	A
3	(3)	A	A
1	(4)	$A \to C$	1 $\wedge$E
3; 1	(5)	C	3, 4 MP
6	(6)	B	A
1	(7)	$B \to C$	1 $\wedge$E
6; 1	(8)	C	6, 7 MP
2; 1	(9)	C	2, 3, 5, 6, 8 $\vee$E
1	(10)	$A \vee B \to C$	2, 9 CP

Here we write each line with a line number (in parentheses), with the assumptions to the left of the line number and the formula to the right of the line number. The assumptions are represented by the line number on which the corresponding formula occurs. On the right of each formula is an annotation that indicates the rule used to deduce that formula, and the previous lines (if any) from which it follows. A formula is a *theorem* of **DW** just when there is a proof that $0 \Vdash A$.

No form of contraction is derivable in **DW**. For example, we cannot prove $A \wedge (A \to B) \to B$. Any proof of *pseudo modus ponens* would have to be something like this:

1	(1)	$A \wedge (A \to B)$	A
1	(2)	A	1 $\wedge$E
1	(3)	$A \to B$	1 $\wedge$E
1; 1	(4)	B	2, 3 MP
1	(5)	B	4 **WI**
1; 0	(6)	B	5 0-right
0	(7)	$A \wedge (A \to B) \to B$	1, 6 CP

But the structural rule **WI** (which warrants the move from $X; X \Vdash A$ to $X \Vdash A$) is not permitted in **DW**, so we cannot prove *pseudo modus ponens*. Later, we will show that the other contraction related inferences are also not valid in **DW**.

However, **DW** is weak. It validates very few inferences involving the conditional. It is possible to strengthen the logic a great deal without admitting contraction related rules such as **WI** by adding properties to intensional bunching. Sensible ones seem to be the following[2]

$$\begin{array}{lrcl}
\mathbf{B} & (X;Y);Z & \Longleftarrow & X;(Y;Z) \\
\mathbf{B}' & (X;Z);Y & \Longleftarrow & X;(Y;Z) \\
\mathbf{C} & X;(Y;Z) & \Longleftarrow & Y;(X;Z) \\
\mathbf{K} & X & \Longleftarrow & Y;X \\
\mathbf{C}'' & 0;X & \Longleftarrow & X
\end{array}$$

Add **B** and **B′** to **DW** to get the stronger logic **TW**. This gives the conditional two transitivity properties. **B** gives us *prefixing*; $(A \to B) \to \big((C \to A) \to (C \to B)\big)$.

1	(1)	$A \to B$	A
2	(2)	$C \to A$	A
3	(3)	C	A
$3;2$	(4)	A	2, 3 MP
$(3;2);1$	(5)	B	1, 4 MP
$3;(2;1)$	(6)	B	5 **B**
$2;1$	(7)	$C \to B$	3, 6 CP
1	(8)	$(C \to A) \to (C \to B)$	2, 7 CP
$1;0$	(9)	$(C \to A) \to (C \to B)$	8 0-right
0	(10)	$(A \to B) \to \big((C \to A) \to (C \to B)\big)$	1, 9 CP

And **B′** gives *suffixing*; $(A \to B) \to \big((B \to C) \to (A \to C)\big)$

1	(1)	$A \to B$	A
2	(2)	$B \to C$	A
3	(3)	A	A
$3;1$	(4)	B	1, 3 MP
$(3;1);2$	(5)	C	2, 4 MP
$3;(2;1)$	(6)	C	5 **B′**
$2;1$	(7)	$A \to C$	3, 6 CP
1	(8)	$(B \to C) \to (A \to C)$	2, 7 CP
$1;0$	(9)	$(B \to C) \to (A \to C)$	8 0-right
0	(10)	$(A \to B) \to \big((B \to C) \to (A \to C)\big)$	1, 9 CP

These two postulates stand or fall together in some contexts. In the presence of a negation like ours the rules are interderivable.[3] For example, you can use **B′** to derive prefixing:

1	(1)	$A \to B$	A
2	(2)	$C \to A$	A
3	(3)	$\sim B$	A
3; 1	(4)	$\sim A$	1, 3 MT
(3; 1); 2	(5)	$\sim C$	2, 4 MT
3; (2; 1)	(6)	$\sim C$	5 $\mathbf{B}'$
7	(7)	C	A
7	(8)	$\sim\sim C$	7 DNI
7; (2; 1)	(9)	$\sim\sim B$	6, 8 MT
7; (2; 1)	(10)	B	9 DNE
2; 1	(11)	$C \to B$	7, 10 CP
1	(12)	$(C \to A) \to (C \to B)$	2, 11 CP
1; 0	(13)	$(C \to A) \to (C \to B)$	12 0-right
0	(14)	$(A \to B) \to ((C \to A) \to (C \to B))$	1, 13 CP

Similarly **B** gives us suffixing. We will show later that given a negation like ours, **B** and $\mathbf{B}'$ amount to exactly the same thing. If we don't have negation, we use both structural rules to get the full strength of the logic **TW**.

This logic is known for the famous **P**—**W** problem. It was conjectured by Anderson and Belnap that in the pure implicational fragment of **TW** (given by deleting the rules for any connective other than the conditional) $A \to B$ and $B \to A$ are both provable only when A and B are the same formula [2]. This was proved in Errol Martin's doctoral thesis of 1978 [78, 79].

To get the stronger logic **EW** from **TW**, add $\mathbf{C}''$. Then $0 \Vdash (t \to A) \to A$.

1	(1)	$t \to A$	A
0	(2)	t	tI
0; 1	(3)	A	1, 2 MP
1	(4)	A	3 $\mathbf{C}''$
1; 0	(5)	A	4 0-right
0	(6)	$(t \to A) \to A$	1, 5 CP

If you read $t \to A$ as 'A is necessary' (it means that A follows from logic) then the axiom $(t \to A) \to A$ is quite appropriate. The contraction-added counterpart **E** was proposed by Anderson and Belnap to give an account of relevant entailment. This was given some credence by the result that the connective $\Rightarrow$, given by taking $A \Rightarrow B$ to be $A \wedge t \to B$ turns out to be an have the properties of entailment in **S4**.

Excursus: It is an interesting exercise to show that adding the dual of $\mathbf{C}''$, $(X \Longleftarrow 0; X)$ leads to the provability of $A \to (t \to A)$. This means that if we 0 to be a left identity for the semicolon as well as a right identity, then we have $A \leftrightarrow (t \to A)$. Given this axiom it is no longer plausible to interpret $t \to A$ as 'necessarily A.' Instead, it is simply 'A is true.' This is not because the meaning of t has changed. It is due to a change in the properties of the conditional. □

To get the stronger logic **C** add the postulate **C** to **DW**. (This logic is traditionally called **R**−**W** or **RW**, because of its relationship with the relevant logic **R**. However, it is so important that it deserves a name that reflects its identity. It is the logic of the postulate **C**, so that is the name given by Slaney [151], and that is the one we will use.) **C** gives the commutativity of intensional bunching, as we get $X; (Y; 0) \Longleftarrow Y; (X; 0)$, which is simply $X; Y \Longleftarrow Y; X$. Then applying commutativity to the rule **C** itself gives the two postulates **B** and **B′**, as well as **C″** and their duals. (So in **C**, 0 is both a left and a right identity for the semicolon.)

The rule **C** is naturally related to the axiom of *permutation*, $\big(A \to (B \to C)\big) \to \big(B \to (A \to C)\big)$.

1	(1)	$A \to (B \to C)$	A
2	(2)	B	A
3	(3)	A	A
3; 1	(4)	$B \to C$	1, 3 MP
2; (3; 1)	(5)	C	2, 4 MP
3; (2; 1)	(6)	C	5 **C**
2; 1	(7)	$A \to C$	3, 6 CP
1	(8)	$B \to (A \to C)$	2, 7 CP
1; 0	(9)	$B \to (A \to C)$	8 0-right
0	(10)	$\big(A \to (B \to C)\big) \to \big(B \to (A \to C)\big)$	1, 9 CP

C is a much stronger logic than any below it. It collapses any modal distinctions, by allowing premises to permute and associate arbitrarily. It keeps account of repetition and use of premises. It is both contraction-free and relevant — in the sense that $A \to B$ is a theorem only when A and B share a propositional variable (or t is involved somewhere). Adding contraction gives the relevant logic **R**, much studied by many.

For **CK** add **K** to **C**. This makes the logic 'irrelevant' in that $A \to (B \to A)$ (called *weakening*, a notorious non-theorem of **R** and all relevant logics) is now provable.

1	(1)	A	A
2	(2)	B	A
2; 1	(3)	A	1 **K**
1	(4)	$B \to A$	2, 3 CP
0; 1	(5)	$B \to A$	4 0-right
0	(6)	$A \to (B \to A)$	1, 5 CP

CK is a very strong logic. It is just a little weaker than Łukasiewicz's infinitely valued logic $\mathbf{Ł}_\infty$, as we shall see later. Premises can associate, commute, and be weakened arbitrarily. Only contraction is prohibited. Because of this, it is an ideal testbed for contraction-free theories, in that it admits almost anything that has been proposed as a valid first-order inference, except for those related to contraction. If we take it that contraction is at root the problem in a particular theory, then successfully formulating that theory in **CK** goes a long way to showing that contraction is, in fact, the problem.

This completes the (rather long) definition of the natural deduction systems. We will call each of the logics **DW**, **TW**, **EW**, **C** and **CK** *our favourite logics.*

Sometimes we will be interested in systems that are very much like some of our favourite logics — but which leave out a few of the connectives of the full systems. If $\mathbf{X}$ is a logic, we let $\mathbf{X}_C$ be the logic on the set C of connectives and quantifiers.

Definition 2.2 For some set C of connectives and quantifiers, the logic $\mathbf{X}_C$ is given by using the same structural rules as $\mathbf{X}$, and the logical rules applying to the connectives in the set C.

So, $\mathbf{X}_{\wedge\vee\circ\rightarrow\forall\exists t}$ is the logic $\mathbf{X}$ without negation. We write $\mathbf{X}^+$ for $\mathbf{X}_{\wedge\vee\circ\rightarrow\forall\exists t}$, the positive logic. However, we often drop subscripts and write $\mathbf{X}$ for one of $\mathbf{X}$'s fragments. The context makes it clear what we mean. In Chapter 3 and Chapter 7 we shall show for many choices of C and D, $\mathbf{X}_C$ conservatively extends $\mathbf{X}_D$ where $\mathbf{X}$ is any of our favourite logics. This means that if we could prove something in $\mathbf{X}_D$ in the vocabulary of $\mathbf{X}_C$ then it is provable in $\mathbf{X}_C$ alone. This means that each of the connective rules characterises the logical properties of that connective — we don't have to rely on any other connectives to give us the facts about $\rightarrow$, for example. If we are interested in proving something in some class of connectives, the conservative extension result shows that we need not go outside that set of connectives to prove it.

EXCURSUS: This is *not* the case with classical logic $\mathbf{K}$, in the system we have given. (You can get classical logic by adding **WI** to **CK**. Or equivalently, by identifying intensional and extensional bunching.) $\mathbf{K}_{\rightarrow}$ is not the conditional fragment of classical logic — it is the positive fragment of *intuitionistic* logic. Peirce's law ($0 \Vdash ((A \rightarrow B) \rightarrow A) \rightarrow A$) is not provable in $\mathbf{K}_{\rightarrow}$ but it is provable in $\mathbf{K}_{\rightarrow\sim}$. So the logical rules for $\rightarrow$ in classical logic do not characterise the material conditional. Extra facts (here, to do with negation) are necessary to derive all of its properties.

Some take it that this has some philosophical significance. The introduction and elimination rules are said to constitute the *meaning* of a logical connective. If this is the case, conservative extension is a desirable thing, because we would like the meaning of a connective to determine its behaviour. If these considerations are important (and here is not the place to discuss this) then our favourite logics come up well.

However, they do not come up perfectly. Adding negation does not conservatively extend the positive logics *with quantification.* Because every negation-free rule is valid intuitionistically, so in the positive logics we cannot prove intuitionistically invalid principles such as

$$\forall x(A \vee B) \Vdash \forall x A \vee \exists x B$$

in the positive systems. However, they do come out as provable in the presence of negation. □

Before moving on to the Hilbert-style presentation of our logics, it is important to discuss a different concept of validity to that encoded by $\Vdash$, and to prove some more principles in our systems. According to contraction-free logics $A \wedge (A \rightarrow B) \Vdash B$ fails. The obvious attempt at proof proceeds as follows:

1	(1)	$A \wedge (A \to B)$	A
1	(2)	A	$1 \wedge$E
1	(3)	$A \to B$	$1 \wedge$E
$1;1$	(4)	B	$2,3$ MP

And it is stuck there. The most we can prove is

$$A \wedge (A \to B); A \wedge (A \to B) \Vdash B$$

which records the fact that to deduce B from $A \wedge (A \to B)$ we must use the premise *twice*. Once to get the antecedent A, and a second time to licence the deduction.

This leaves us in a quandry of sorts. In some sense B does follow from $A \wedge (A \to B)$. For example, if $0 \Vdash A \wedge (A \to B)$ then we must have $0 \Vdash B$. This is because 0 is a special bunch — it contracts, because $0; 0$ can be replaced by 0 in any context. To make use of this kind of validity, we will define $\vdash$ to encode it.

Definition 2.3 $\Sigma \vdash A$ means that if we assume that $0 \Vdash B$ for every $B \in \Sigma$, then $0 \Vdash A$ too. To prove that $\Sigma \vdash A$ in the natural deduction formalism, you need only make the hypothesis that each element of Σ is true at 0, and proceed from there, to show that B is true at 0 too.

This definition is a little odd, but it makes sense, given the definition a Hilbert-style proofs (which we shall come to). Recall that a Hilbert proof from a set of assumptions is simply a list, with the conclusion as the final element, with every element of the list either an axiom or an assumption, or following from earlier elements of the list by one of the rules. This approach makes no distinction between axioms and assumptions. Our definition of $\vdash$ is similar. We give all of our assumptions the status of logical truths, and we reason from there. For example, we will show that $A \to B \vdash (C \to A) \to (C \to B)$ in **DW**.

0	(1)	$A \to B$	Hyp
2	(2)	$C \to A$	A
3	(3)	C	A
$3;2$	(4)	A	$2,3$ MP
$(3;2);0$	(5)	B	$1,4$ MP
$3;2$	(6)	B	5 0-right
2	(7)	$C \to B$	$3,6$ CP
$2;0$	(8)	$C \to B$	5 0-right
0	(9)	$(C \to A) \to (C \to B)$	2 CP

Similarly, in **EW** we have $A \vdash (A \to B) \to B$.

0	(1)	A	Hyp
2	(2)	$A \to B$	A
$0;2$	(3)	B	$1,2$ MP
2	(4)	B	3 $\mathbf{C}''$
$2;0$	(5)	B	4 0-right
0	(6)	$(A \to B) \to B$	$2,5$ CP

It is simple to show that if $\Sigma \Vdash A$ then $\Sigma \vdash A$ (for finite sets Σ, and where $\{B_1, \ldots, B_n\} \Vdash A$ is read as $B_1, \ldots, B_n \Vdash A$) but the converse doesn't hold. Because of this, we will call the notion encoded by $\Vdash$ *strong validity* and that encoded by $\vdash$ *weak validity*.[4]

We must make a policy about free variables in things that are hypothesised for the side conditions in the quantifier rules to make sense. We take it that if we hypothesise A, and x is free in A, then x is free in 0 for the extent of the proof. It follows that $A(x) \vdash A(x)$, but $A(x) \nvdash \forall x A(x)$ and $A(x) \nvdash A(y)$ (where y is distinct from x). This seems sensible, under one interpretation of free variables. That is, $A(x)$ with free x only makes sense under an assignment to the variable x. The argument from Σ to A is valid only when under each assignment to the free variables in Σ and A, the resulting argument is valid. So, $A(x) \vdash A(y)$ is not valid in general, because we may assign different values to x and y.

Now we can prove some results about weak validity.

Lemma 2.1 *If* $\Sigma \vdash A$ *and* $\Delta \cup \{A\} \vdash B$ *then* $\Delta \cup \Sigma \vdash A$.

Proof: Take the natural deduction proof that demonstrates that $\Delta \cup \{A\} \vdash B$. Replace the line (or lines) that hypothesise A with the proof of A from the hypothesis that Σ. Then the resulting proof demonstrates B under the hypothesis that $\Sigma \cup \Delta$. $\triangleleft$

Given that contraction (or the lack thereof) is the focus of this work, it is important to get some kind of language with which to describe repetitions of premises. As the semicolon is modelled by fusion, we want to be able to talk about repeated fusions.

Definition 2.4 In our language we take A^n to be a shorthand for the n-fold fusion of A with itself — associated to the right. So $A^0 = t$, and $A^{n+1} = A \circ A^n$.

Lemma 2.2 $A \vdash A^n$ *for each* n.

Proof: Clearly $A \vdash t$. So our result holds for $n = 0$. If our result holds for n, then the proof

0	(1)	A	Hyp
0	(2)	A^n	1 Induction hypothesis
$0;0$	(3)	$A \circ A^n$	$1, 2 \circ$I
0	(4)	A^{n+1}	3 0-right

shows how to derive the result for $n + 1$ as well. $\triangleleft$

This shows how weak validity admits contractions. In all of our favourite logics, we cannot prove that $A \Vdash A^n$ because this would be to endorse **WI**. However, $A \vdash A^n$, since we may use hypotheses any number of times.

Corollary $\{A, A^n \to B\} \vdash B$

Proof: We have $A \vdash A^n$, so $\{A^n, A^n \to B\} \vdash B$ gives $\{A, A^n \to B\} \vdash B$ as desired. $\triangleleft$

These results will be useful in the following section.

2.3 Hilbert Systems

For many, the Lemmon-style natural deduction formulation of a logic will not be familiar. Also, for some uses (such as soundness and completeness proofs) it may not be the most useful presentation of a logic. For these reasons it is wise to give an alternate description of our logics. In this case, it is a Hilbert-style axiomatisation. The axioms are the formulae of the following form. Firstly, the *propositional axioms*.

Identity	$A \to A$
$\wedge$ *Elimination*	$A \wedge B \to A$, $A \wedge B \to B$
$\wedge$ *Introduction*$_1$	$(A \to B) \wedge (A \to C) \to (A \to B \wedge C)$
$\vee$ *Elimination*	$(A \to C) \wedge (B \to C) \to (A \vee B \to C)$
$\vee$ *Introduction*	$A \to A \vee B$, $B \to A \vee B$
Distribution	$A \wedge (B \vee C) \to (A \wedge B) \vee (A \wedge C)$
Contraposition	$(A \to {\sim} B) \to (B \to {\sim} A)$
Double Negation	${\sim}{\sim} A \to A$

These axioms give the conjunction/disjunction fragment of the logic the structure of a distributive lattice. The negation is then forced to be a de Morgan negation on that lattice. See Chapter 3 for a fuller explanation of these terms. The *quantifier axioms* are also quite simple.

$\forall$ *Elimination*	$\forall x A(x) \to A(t)$	t free for x in $A(x)$
$\forall$ *Introduction*	$A \to \forall x A$	x not free in A
$\forall \to$ *Distribution*	$\forall x(A \to B) \to (A \to \forall x B)$	x not free in A
$\forall\vee$ *Distribution*	$\forall x(A \vee B) \to A \vee \forall x B$	x not free in A
$\exists$ *Introduction*	$A(t) \to \exists x A(x)$	t free for x in $A(x)$
$\exists$ *Distribution*	$\forall x(A \to B) \to (\exists x A \to B)$	x not free in B

For each axiom A we take $\forall x_1 \cdots \forall x_n A$ to be an axiom too. (This does away with the need any kind of rule for universal quantification. This is good, because according to our natural deduction system, $A \vdash \forall x A$ fails.) The rules we do need are:

$\wedge$ *Introduction*$_2$	$A, B \vdash A \wedge B$
Modus Ponens	$A \to B, A \vdash B$
Rule Prefixing	$A \to B \vdash (C \to A) \to (C \to B)$
Rule Suffixing	$A \to B \vdash (B \to C) \to (A \to C)$
t *Definition*	$t \to A \dashv\vdash A$
$\circ$ *Definition*	$A \circ B \to C \dashv\vdash B \to (A \to C)$

These axioms and rules are sufficient for **DW**. (We will use these axioms and rules often, and mention them by name. It is important to remember them.)

EXCURSUS: These rules make the behaviour of $\vdash$ obvious. It is simple to prove in the natural deduction system that $A \vdash t \to A$. However, where t stands for 'logic,' as it does in our systems, and where the implication has modal force, as it does at least in **DW**, **TW** and **EW**, it is odd to have a deduction that proclaims that $t \to A$ (which is a good candidate for $\Box A$) follows from A. This threatens to trivialise any account of modality.

However, the threat is an empty one. The crux of the matter is the interpretation of $\vdash$, which as we have noted, takes the hypotheses to be true as a matter of logic. If we hypothesise A in this manner, it is not surprising that $t \to A$ would follow. In Section 2.10 we will elaborate on this matter further, to indicate notions of validity less stringent than $\Vdash$, but for which $\Box A$ does not follow from A. $\Box$

The logic **TW** is given by adding *Axiom Prefixing* and *Axiom Suffixing* to **DW**;

$$\textit{Axiom Prefixing} \quad (A \to B) \to \big((C \to A) \to (C \to B)\big)$$
$$\textit{Axiom Suffixing} \quad (A \to B) \to \big((B \to C) \to (A \to C)\big)$$

EW is given by adding *Restricted Assertion* (either the axiom, or the rule) to **TW**.

$$\textit{Restricted Assertion} \quad (t \to A) \to A, \quad A \vdash A \to (A \to B)$$

C is obtained by adding *Permutation* to **DW**;

$$\textit{Permutation} \quad \big(A \to (B \to C)\big) \to \big(B \to (A \to C)\big)$$

and **CK** is obtained by adding *Weakening* to **C**.

$$\textit{Weakening} \quad A \to (B \to A)$$

The Hilbert-style system thus encodes weak validity — according to a Hilbert system, $\Sigma \vdash A$ if and only if there is a list of formulae ending in A, such that each element of the list follows from earlier elements by rules from the Hilbert-system in question, or is either an element of Σ or an axiom. One obvious result we require is the theorem:

THEOREM 2.3 *In* **DW**, *if there is a Hilbert style proof that $\Sigma \vdash A$, then there is also a natural deduction proof that $\Sigma \vdash A$*

Proof: This is a tedious case of proving in each of the Hilbert axioms in the natural deduction system, and then showing that the rules preserve truth-at-0. Two of the more difficult cases (for $\lor$ Elimination and the prefixing rule) have been done in the previous section. We will work out one more case. Assume that x is not free in B, that y is a variable distinct from x, and that in $A(x)$, all free occurrences of x are indicated. Then we can prove $\exists$ Elimination as follows:

1	(1)	$\forall x(A(x) \to B)$	A
2	(2)	$\exists \nu A(x)$	A
3	(3)	$A(y)$	A
1	(4)	$A(y) \to B$	1 $\forall$E
3; 1	(5)	B	3, 4 MP
2; 1	(6)	B	2, 3, 5 $\exists$E
0	(7)	$\forall x(A(x) \to B) \to (\exists x A(x) \to B)$	Fiddling

The other cases are left to the enthusiastic reader. ◁

To show the equivalence in the other direction, it is sufficient to interpret each of the natural deduction rules as truth preserving rules in the Hilbert system. Firstly we need to be able to interpret the bunches of formulae as formulae.

Definition 2.5 If X is a bunch of formulae, then the formula $\mathcal{I}(X)$ is defined inductively as follows:

$$\mathcal{I}(A) = A \qquad \mathcal{I}(0) = t \qquad \mathcal{I}(X, Y) = \mathcal{I}(X) \wedge \mathcal{I}(Y) \qquad \mathcal{I}(X; Y) = \mathcal{I}(X) \circ \mathcal{I}(Y)$$

$\mathcal{I}(X)$ is called the *debunchification* of X. It is clear that a debunchification is built up from formulae by the application of conjunction and fusion. Such a formula is affectionately called a *confusion*. Using debunchifications, we can model the sequents from the natural deduction system as formulae that can be operated on in the Hilbert system.

These structures are reasonably well behaved. In particular, it is simple to derive the following fact.

Lemma 2.4 *If $Y(X)$ is a bunch in which X appears as a particular sub-bunch then $\mathcal{I}\big(Y(X)\big) = \mathcal{I}\big(Y(\mathcal{I}(X))\big)$.*

Proof: A simple matter of applying the definitions. ◁

Another indication of $\mathcal{I}$'s good behaviour is the following fact.

Lemma 2.5 *In the Hilbert system, $A \to B \vdash \mathcal{I}\big(Y(A)\big) \to \mathcal{I}\big(Y(B)\big)$.*

Proof: By induction on the formation of $Y(A)$. It relies on the simple facts:

$$A \to B \vdash C \wedge A \to C \wedge B \qquad A \to B \vdash C \circ A \to C \circ B \qquad A \to B \vdash A \circ C \to B \circ C$$

which may be verified in the Hilbert system. The details are left to the reader. ◁

This is what we need to deal with proving the natural deduction rules in the Hilbert system. The next two results that help us prove the required properties of disjunction and the existential quantifier.

Lemma 2.6 *In the Hilbert system we have $\vdash \mathcal{I}\big(Y(A \vee B)\big) \to \mathcal{I}\big(Y(A)\big) \vee \mathcal{I}\big(Y(B)\big)$ and $\vdash \mathcal{I}\big(Y(\exists x A)\big) \to \exists x \mathcal{I}\big(Y(A)\big)$ if x is not free elsewhere in $Y(\exists x A)$.*

Proof: For disjunction it is an easy induction on the formation of $Y(A \vee B)$, where we need

$$\vdash A \wedge (B \vee C) \to (A \wedge B) \vee (A \wedge C)$$

$$\vdash A \circ (B \vee C) \to (A \circ B) \vee (A \circ C)$$

$$\vdash (B \vee C) \circ A \to (B \circ A) \vee (C \circ A)$$

as the base cases. These are tedious but not difficult to verify using the Hilbert system. The induction step is then taken care of by the previous lemma. For the existential quantifier, the base cases are instead

$$\vdash A \wedge \exists x B \to \exists x (A \wedge B)$$

$$\vdash A \circ \exists x B \to \exists x (A \circ B)$$

$$\vdash (\exists x B) \circ A \to \exists x (B \circ A)$$

where x is not free in B. These are also tedious to prove in the Hilbert system, and a little more difficult, so we'll prove one (the last) and leave the rest for the reader. First, by the fusion-defining rules it is sufficient to prove

$$A \to \big(\exists x B \to \exists x(B \circ A)\big) \qquad (\alpha)$$

We already have $A \to (B \to B \circ A)$ (as a simple consequence of $\circ$ *Definition*) and as x is free in A, we have $A \to \forall x(B \to B \circ A)$. Now as $\vdash B \circ A \to \exists x(B \circ A)$ (as x is free for x wherever it appears!) we have $\forall x(B \to B \circ A) \to \forall x\big(B \to \exists x(B \circ A)\big)$ (doing a little rule prefixing and suffixing). But now x is not free in $\exists x(B \circ A)$ so we can apply $\exists$ Distribution to give $\forall x(B \to B \circ A) \to \big(\exists x B \to \exists x(B \circ A)\big)$ which, with what we already have gives us (α) as desired. ◁

Now we have enough to verify the converse implication to that in Theorem 2.3.

THEOREM 2.7 *In* **DW**, *if there is a natural deduction proof that* $\Sigma \vdash A$, *then there is also a Hilbert proof that* $\Sigma \vdash A$

Proof: This is a matter of verifying each of the natural deduction rules in the their interpretation in the Hilbert system. The rule of assumptions is simple. Any *special* assumption of an element B of Σ with assumption 0 amounts to assuming $t \to B$, which is equivalent to assuming B by the definition of t. Verifying each of the rules is tedious. The structural rules follow from properties of fusion and conjunction. We will work three of the introduction and elimination rules and leave the rest to the committed.

For MP, we have $\mathcal{I}(X) \to (A \to B)$ and $\mathcal{I}(Y) \to A$ as premises. We wish to derive $\mathcal{I}(X;Y) \to B$, or the equivalent $\mathcal{I}(X) \to (\mathcal{I}(Y) \to B)$. We have $(A \to B) \to (\mathcal{I}(Y) \to B)$ by prefixing and our second premise. Transitivity from the first premise and this, gives what we want.

For $\vee$E, the premises are interpreted as $\mathcal{I}(X) \to A \vee B$, $\mathcal{I}(Y(A)) \to C$ and $\mathcal{I}(Y(B)) \to C$. From the first premise and Lemma 2.6 we have $\mathcal{I}(Y(X)) \to \mathcal{I}(Y(A \vee B))$. But we have $\vdash \mathcal{I}(Y(A \vee B)) \to \mathcal{I}(Y(A)) \vee \mathcal{I}(Y(B))$, so we derive $\mathcal{I}(Y(X)) \to \mathcal{I}(Y(A)) \vee \mathcal{I}(Y(B))$. The other two premises give $\mathcal{I}(Y(A)) \vee \mathcal{I}(Y(B)) \to C$, and so we derive $\mathcal{I}(Y(X)) \to C$, which is what we wished.

For $\exists$E, the premises are mapped to $\mathcal{I}(X) \to \exists x A$ and $\mathcal{I}(Y(A)) \to B$ where x is not free in $Y(B)$. This means that we have $\forall x(\mathcal{I}(Y(A)) \to B)$ and hence $\exists x \mathcal{I}(Y(A)) \to B$ as x is not free in B. Since x is not free in $\mathcal{I}(Y(-))$ (remainder of $\mathcal{I}(Y(A))$ after deleting A) we can apply Lemma 2.6 to give $\mathcal{I}(Y(\exists x A)) \to B$ which with $\mathcal{I}(X) \to \exists x A$ gives $\mathcal{I}(Y(X)) \to B$ as desired. ◁

The only other thing to do is to show that the Hilbert-style axiomatisation of **TW**, **EW**, **C** and **CK** are equivalent to their natural deduction formulations. We wish to prove the following theorem

THEOREM 2.8 *For* **TW**, **EW**, **C** *and* **CK** *there is a natural deduction proof that* $\Sigma \vdash A$, *if and only if there is also a Hilbert proof of* $\Sigma \vdash A$.

Proof: We have shown that if $\Sigma \vdash A$ is provable in the Hilbert system, then it is provable in the natural deduction system, as we have shown that the Hilbert-style axioms are provable in the natural deduction system.

For the other direction, we must show that the structural rules added to the natural deduction system to give an extension is also provable in the Hilbert system, under interpretation. We will prove the case for **C** and leave the rest to the reader. For **C** we must show that the structural rule $X; (Y; Z) \Longleftarrow Y; (X; Z)$ is provable under interpretation. This amounts to the provability of $\mathcal{I}\big(Y; (X; Z)\big) \to \mathcal{I}\big(X; (Y; Z)\big)$ which in turn amounts to that of $B \circ (A \circ C) \to A \circ (B \circ C)$. To do this, note that the Hilbert rule $\circ$ *Definition* applied twice gives us

$$\vdash C \to \big(B \to \big(A \to A \circ (B \circ C)\big)\big)$$

and by permutation and prefixing we have

$$\vdash C \to \big(A \to \big(B \to A \circ (B \circ C)\big)\big)$$

which by reapplying the fusion definition rules give $B \circ (A \circ C) \to A \circ (B \circ C)$ as we desired.

All other rules are proved in exactly the same way. By unpacking the fusion-related natural deduction rule into an implication, applying the implication-related axiom from the logic, and then packing it back up again. $\triangleleft$

Similarly, if $\Sigma \vdash A$ using **WI** in a natural deduction system, then there is a proof of this fact in the relevant Hilbert system using the *pseudo modus ponens* axiom

$$A \wedge (A \to B) \to B \qquad \textbf{PMP}$$

as another axiom. And *vice versa.*

2.4 Converse Implication

In certain applications we may be interested in the connective $\leftarrow$ which is like $\to$ but 'goes in the other direction.' It is satisfies the following natural deduction rules:

$$\frac{X \Vdash B \leftarrow A \quad Y \Vdash A}{X; Y \Vdash B}\ \mathrm{MP}_{\leftarrow} \qquad \frac{X; A \Vdash B}{X \Vdash B \leftarrow A}\ \mathrm{CP}_{\leftarrow}$$

The corresponding Hilbert-style axiomatisation is

$$A \circ B \to C \dashv\vdash A \to (C \leftarrow B)$$

It is a rather simple exercise to show that these two formalisations are equivalent. Converse implication is of interest because of its application to the analysis of sentence structure in the Lambek calculus (see Chapter 10). However, it is also interesting because it gives us a way to model the dual of condition **B**, which we shall call $\mathbf{B}^{dual}$.

$$\mathbf{B}^{dual} \quad X; (Y; Z) \Longleftarrow (X; Y); Z$$

It is not too difficult to prove that under the Hilbert-system this is equivalent to

$$(A \leftarrow B) \to ((A \leftarrow C) \leftarrow (B \leftarrow C))$$

which can be proved as follows

1	(1)	$A \leftarrow B$	A
2	(2)	$B \leftarrow C$	A
3	(3)	C	A
$2;3$	(4)	B	$2,3\,\mathrm{MP}_{\leftarrow}$
$1;(2;3)$	(5)	A	$1,4\,\mathrm{MP}_{\leftarrow}$
$(1;2);3$	(6)	A	$5\,\mathbf{B}^{dual}$
$1;2$	(7)	$A \leftarrow C$	$3,6\,\mathrm{CP}_{\leftarrow}$
1	(8)	$(A \leftarrow C) \leftarrow (B \leftarrow C)$	$2,7\,\mathrm{CP}_{\leftarrow}$
$1;0$	(9)	$(A \leftarrow C) \leftarrow (B \leftarrow C)$	8 0-right
0	(10)	$(A \leftarrow B) \rightarrow ((A \leftarrow C) \leftarrow (B \leftarrow C))$	$1,9\,\mathrm{CP}$

The converse proof (showing that $X;(Y;Z) \Longleftarrow (X;Y);Z$ comes out under interpretation) is not difficult either.

In positive logics like $\mathbf{DW}^+$ we can add $\mathbf{B}$ without $\mathbf{B}'$ being a consequence. This means that we can add both $\mathbf{B}$ and $\mathbf{B}^{dual}$ to $\mathbf{DW}^+$ to get a system in which intensional bunching is truly associative. We call this logic $\mathbf{L}^+$, as its implicational fragment $\mathbf{L}_{\rightarrow\leftarrow}$ is the Lambek calculus. Adding $\mathbf{C}''$ and its dual gives the Lambek calculus with identity, $\mathbf{LI}^+$. (See Chapter 10 for details Lambek's calculus.)

If we add the permutation postulate $\mathbf{C}$ to a logic, then converse implication and ordinary implication collapse. In this sense, $\mathbf{L}^+$ and $\mathbf{LI}^+$ are subsystems of $\mathbf{C}^+$. $\mathbf{L}^+$ and $\mathbf{LI}^+$ are two more of our favourite logics.

2.5 Contraction and its Cousins

Many classically valid principles turn out to be related to contraction. As we saw, **WI** amounts to *pseudo modus ponens*. But any principle that replaces an assumption bunch with another bunch that fewer intensional repetitions of some sub-bunch is also contraction related. There are three more standard rules that satisfy this condition.

W	$X;(X;Y)$	$\Longleftarrow$	$X;Y$
CSyll	$(X;Y);Y$	$\Longleftarrow$	$X;Y$
S	$(X;Y);(X;Z)$	$\Longleftarrow$	$X;(Y;Z)$

In the context of **C** adding any of these has exactly the same effect as adding **WI**. But they diverge in weaker systems.

In all systems, **W** warrants $A \rightarrow (A \rightarrow B) \Vdash A \rightarrow B$, as follows:

1	(1)	$A \rightarrow (A \rightarrow B)$	A
2	(2)	A	A
$2;1$	(3)	$A \rightarrow B$	$1,2\,\mathrm{MP}$
$2;(2;1)$	(4)	B	$1,3\,\mathrm{MP}$
$2;1$	(5)	B	$4\,\mathbf{W}$
1	(6)	$A \rightarrow B$	$1,5\,\mathrm{CP}$

Clearly, **W** is stronger than **WI**. At the level or rules, **W** gives $X;X \Longleftrightarrow X;(X;0) \Longleftarrow X;0 \Longleftrightarrow X$. In the Hilbert system, $\vdash (A \rightarrow B) \rightarrow (A \rightarrow B)$ gives

$$\vdash A \wedge (A \rightarrow B) \rightarrow (A \wedge (A \rightarrow B) \rightarrow B)$$

strengthening two antecedents. Then, $\vdash A \wedge (A \to B) \to B$ follows by applying **W**.

Priest, in "Sense, Entailment and *Modus Ponens*" [122], argues that $(A \to B) \to (A \to B)$ is the *real* axiomatic form of *modus ponens*. These considerations bring this out. Note that if read correctly: "If A implies B then if A [pause] then B" it sounds just right! Applying **WI** to get the imposter is unnecessary, and unwarranted.

In all systems, **CSyll** gives $(A \to B) \wedge (B \to C) \Vdash A \to C$ (*conjunctive syllogism*) as follows:

1	(1)	$(A \to B) \wedge (B \to C)$	A
1	(2)	$A \to B$	$1 \wedge$E
1	(3)	$B \to C$	$1 \wedge$E
4	(4)	A	A
$4;1$	(5)	B	$2,4$ MP
$(4;1);1$	(6)	C	$3,5$ MP
$4;1$	(7)	C	7 **CSyll**
1	(8)	$A \to C$	$4,7$ CP

This shows that in the deduction of conjunctive syllogism, we have collapsed the two applications of the premise (the conjoined conditionals) into one — a paradigmatic instance of contraction.

Finally, the self distribution law $A \to (B \to C) \Vdash (A \to B) \to (A \to C)$ is given by the structural rule **S**.

1	(1)	$A \to (B \to C)$	A
2	(2)	$A \to B$	A
3	(3)	A	A
$3;1$	(4)	$B \to C$	$1,3$ MP
$3;2$	(5)	B	$2,3$ MP
$(3;2);(3;1)$	(6)	C	$4,5$ MP
$3;(2;1)$	(7)	C	6 **S**
$2;1$	(8)	$A \to C$	$3,7$ CP
1	(9)	$(A \to B) \to (A \to C)$	$2,8$ CP

The self distribution postulate collapses two uses of A into one. We use A once to get $B \to C$ and then again to get B. If multiple applications of antecedents are banned, **S** fails.

There are negation postulates that are associated with contraction too, to a slightly lesser extent. Two are excluded middle and *reductio*.

$$\vdash A \vee {\sim} A \qquad A \to {\sim} A \Vdash {\sim} A$$

These are given by the following rules:

$$\frac{A \Vdash {\sim} A}{0 \Vdash {\sim} A}\ \textbf{ExMid} \qquad \frac{A;X \Vdash {\sim} A}{X \Vdash {\sim} A}\ \textbf{Red}$$

by the following proofs. *Reductio* is very simple.

1	(1)	$A \to \sim A$	A
2	(2)	A	A
2; 1	(3)	$\sim A$	1, 2 MP
1	(4)	$\sim A$	2, 3 **Red**

Excluded middle is a little harder to prove, given **ExMid**.

1	(1)	$A \wedge \sim A$	A
1	(2)	A	1 $\wedge$E
1	(3)	$\sim\sim A$	2 DNI
1	(4)	$\sim A$	1 $\wedge$E
1	(5)	$\sim(A \wedge \sim A)$	3, 4 MT
0	(6)	$\sim(A \wedge \sim A)$	1, 5 **ExMid**

This is the significant part of the proof. The rest, to apply de Morgan's law to get the disjunctive form, is tedious.

7	(7)	A	A
7	(8)	$A \vee \sim A$	7 $\vee$I
9	(9)	$\sim(A \vee \sim A)$	A
9	(10)	$\sim A$	8, 9 MT
11	(11)	$\sim A$	A
11	(12)	$A \vee \sim A$	11 $\vee$I
9	(13)	$\sim\sim A$	9, 12 MT
9	(14)	A	13 DNE
9	(15)	$A \wedge \sim A$	10, 14 $\vee$I
0	(16)	$\sim\sim(A \vee \sim A)$	6, 15 MT
0	(17)	$A \vee \sim A$	16 DNE

Now both of these rules are contraction related in some contexts. In the presence of $\mathbf{C}''$ it turns out that **ExMid** is a slightly weaker version of **WI**, because **ExMid** validates the inference from $A; A \Vdash \sim t$ to $A \Vdash \sim t$ — the deduction goes like this. From $A; A \Vdash \sim t$ we get $t; A \Vdash \sim A$ by a contraposition, $\mathbf{C}''$ gives $A \Vdash \sim A$, which gives $t \Vdash \sim A$ by **ExMid**, and hence $A \Vdash \sim t$ by another contraposition. So, in the presence of $\mathbf{C}''$, excluded middle amounts to a limited form of contraction.

Reductio is stronger than excluded middle, and so it is contraction related in these contexts too. The connection with contraction is enough to ensure that excluded middle and *reductio* are absent from our logics. (It must be noted that it is possible to add excluded middle to logics as strong as **C** without complete contraction resulting.[5] But we take this as no recommendation for its admission. As noted in Chapter 1, it's not obvious that we ought to endorse excluded middle, so if we can get away without it, it is sensible to do so.)

2.6 Basic Theorems

There are a number of results in our logics that come up repeatedly, so we will collect them here.

Firstly, it is simple to generalise the results of Lemma 2.5 to show the substitution theorem.

THEOREM 2.9 *In any of our systems, if* $\vdash A \leftrightarrow B$ *then* $\vdash C \leftrightarrow C'$ *where* C' *results from* C *by replacing any number of occurrences of* A *as a subformula by* B*.*

Proof: By induction on the complexity of C. The base case is trivial. The steps are cases like

$$\vdash A \leftrightarrow B \text{ only if } \vdash A \wedge D \leftrightarrow B \wedge D$$
$$\vdash A \leftrightarrow B \text{ only if } \vdash \exists x A \leftrightarrow \exists x B$$

which have been proved before or are trivial. The details are left to the reader. ◁

The quantifier rules turn out to be quite classical.

LEMMA 2.10 *In each of our favourite logics,* $\sim\exists x \sim Ax \Vdash \forall x Ax$ *and* $\exists x \sim Ax \Vdash \sim\forall x Ax$*.*

Proof:

1	(1)	$\sim\exists x \sim Ax$	A
2	(2)	$\sim Aa$	A
2	(3)	$\exists x \sim Ax$	2 ∃I
2	(4)	$\sim\sim\exists x \sim Ax$	3 DNI
1	(5)	$\sim\sim Aa$	1,4 MT
1	(6)	Aa	5 DNE
1	(7)	$\forall x Ax$	6 ∧I

1	(1)	$\exists x \sim Ax$	A
1	(2)	$\sim Aa$	1 ∃E
3	(3)	$\forall x Ax$	A
3	(4)	Aa	3 ∧E
3	(5)	$\sim\sim Aa$	4 DNI
1	(6)	$\sim\forall x Ax$	1,5 MT

◁

In stronger logics, more things become provable. For example, the relationship between fusion and implication becomes tighter in the presence of permutation.

LEMMA 2.11 *In* **C** *and above,* $\vdash A \circ B \leftrightarrow \sim(A \rightarrow \sim B)$.

Proof: It is sufficient to prove that $\vdash A \circ B \rightarrow C$ if and only if $\vdash \sim(A \rightarrow \sim B) \rightarrow C$. But this is simple:

$\vdash \sim(A \rightarrow \sim B) \rightarrow C$	if and only if (contraposing the whole)
$\vdash \sim C \rightarrow (A \rightarrow \sim B)$	if and only if (contraposing inside)
$\vdash \sim C \rightarrow (B \rightarrow \sim A)$	if and only if (permuting the whole)
$\vdash B \rightarrow (\sim C \rightarrow \sim A)$	if and only if (contraposing inside)
$\vdash B \rightarrow (A \rightarrow C)$	if and only if (by the fusion rule)
$\vdash A \circ B \rightarrow C$	as desired.

And from this result, $\vdash A \circ B \rightarrow \sim(A \rightarrow \sim B)$ (letting C be $\sim(A \rightarrow \sim B)$) and $\vdash \sim(A \rightarrow \sim B) \rightarrow A \circ B$ (letting C be $A \circ B$) as desired. ◁

COROLLARY *In* **C** *and above,* $\vdash (A \circ B \rightarrow C) \leftrightarrow \big(B \rightarrow (A \rightarrow C)\big)$

Proof: It is a simple exercise in permutation and contraposition to show that

$$\vdash \big(\sim(A \to \sim B) \to C\big) \leftrightarrow \big(B \to (A \to C)\big)$$

Prove this, apply the lemma and you are done. $\triangleleft$

This is one example of rules in weaker logics becoming theorems in stronger logics; the fusion rule becomes strongly valid, and expressible in a theorem. The same can be seen in **EW**, in which $\vdash (t \to A) \to A$. This was merely the weakly valid rule $t \to A \vdash A$ in weaker logics. Then the rule $A \vdash t \to A$ which is weakly valid in all of our logics only becomes encoded in the theorem $\vdash A \to (t \to A)$ in logics as strong as **C** or **LI**.

A final result will show the difference between **CK** and weaker logics.

LEMMA 2.12 *In* **CK**, $\vdash A \to t$ *for any* A, *and so* $\vdash A \leftrightarrow A \wedge t$ *and* $\vdash A \leftrightarrow (A \leftrightarrow t)$.

Proof: The first half is easy.

0	(1)	t	tI
2	(2)	A	A
2;0	(3)	t	1**K**
0	(4)	$A \to t$	2,3CP

Then $\vdash A \to A$ and $\vdash A \to t$ give $\vdash A \to A \wedge t$, which with $\vdash A \wedge t \to A$ gives the first biconditional.

For the second, $\vdash A \to (t \to A)$ by **K** and **K** also gives $\vdash A \to (A \to t)$ so $\vdash A \to (A \leftrightarrow t)$. Conversely, $\vdash (t \to A) \to A$ so $\vdash (t \leftrightarrow A) \to A$. Conjoining these two gives us the result. $\triangleleft$

Considering propositions as ordered by provable implication (so A is under B just when $\vdash A \to B$) then this result shows that t is at the top. This result is not provable in any other of our favourite logics logic, because if $\vdash A \to t$ in a logic, then a close cousin of **K** is provable.

LEMMA 2.13 *In any logic containing* **DW**, *if* $\vdash A \to t$ *then* $\vdash A \to (B \to B)$.

Proof: If $\vdash A \to t$, then the **DW** theorem $\vdash t \to (B \to B)$ and transitivity gives $\vdash A \to (B \to B)$ as desired. $\triangleleft$

The principle $A \to (B \to B)$ (called $\mathbf{K}'$) is present in only **CK** of all of our favourite logics. Permuting $\mathbf{K}'$ gives **K**, so if $\mathbf{K}'$ were present in **C**, so would **K**, which would strengthen **C** to **CK**.

2.7 Relationships

These logics do not arise out of a historical vacuum. The systems **DW**, **TW**, **EW** and **C** all arose from the study of relevant logics. The most important relevant logic is **R**, which is given by adding **W** (or **WI**) to **C**. (Historically, **C** — more often called **RW** or even **R**−**W** — was defined by removing contraction from **R**.) In **R** fusion is associative, commutative and idempotent — it has all of the features of extensional conjunction except for weakening. So in **R**, you have $A \wedge B \vdash B$ without also having $A \circ B \vdash B$ for all

A and B. This means that $B \vdash A \to B$ fails for many A and B — a crucial condition for the logic to be *relevant*. If $\vdash A \to B$ in the fragment of **R** without t then A and B share some predicate symbols. For $A \to B$ to be a theorem, A has to be somehow *relevant* to B. (If we restore t to the logic, we add that t is relevant to everything, because if $\vdash A$ then $\vdash t \to A$.) This means that the paradoxes of implication, such as $A \to B \vee {\sim} B$ and $A \wedge {\sim} A \to B$ are not theorems of **R**. For more on relevant logics, see the Anderson, Belnap and Dunn's *Entailment* [1, 5] and Dunn's "Relevance Logic and Entailment" [30].

Other logics are similar to our favourites. Linear logic bears a striking resemblance to **C**. The additive and multiplicative (extensional and intensional) fragment of linear logic is given by taking the Hilbert axiomatisation of **C** and dropping the distribution of $\wedge$ over $\vee$ [51]. Equivalently, it can be given by dropping extensional bunching altogether from the natural deduction system, and modelling conjunction and disjunction in a structure-free way by adopting a different conjunction introduction rule.

$$\frac{X \Vdash A \quad X \Vdash B}{X \Vdash A \wedge B}\ \text{variant} \wedge\text{I}$$

Then there is no way to prove distribution (because it essentially requires the contraction and weakening properties of the extensional bunching operator).

Much is made of the 'naturalness' of linear logic, as it is also given by removing the structural rules of weakening and contraction from the standard Gentzen system for classical logic. And as the system is very close to one we work with, we will not disagree with this estimation too strongly. However, in what follows we will concentrate exclusively on systems with distribution of extensional conjunction over disjunction, because they are also very natural. We have argued before that our conjunction and disjunction rules get the logic of conjunction and disjunction right. In the absence of any obvious use for non-distributing conjunction and disjunction (the front-running candidate, quantum logic, is too verificationist for my liking) we will continue to consider the systems with orthodox extensional connectives. We shall see in Chapters 8–10 that our range of systems have many applications, each of which validate distribution. The arguments of this thesis can be seen as not only endorsing contraction-free logics (against those who see no use for them) but also as defending contraction-free logics *with distribution* against those who see worth only in linear logic and its cousins.

The historically earliest contraction-free logics are the many-valued logics due to Łukasiewicz. It is simple to show that contraction fails in each of Łukasiewicz's many valued logics.

> The most important theses of the two-valued calculus which do not hold true for the three- and infinite-valued systems concern certain apagogic inference schemata that have been suspect from time immemorial. For example, the following theses do not hold true in many-valued systems: "CCNppp," "CCpNp Np," "CCpqCCpNqNp," "CCpKqNqNp," "CCpEqNqNp" ...

> There are modes of inference in mathematics, among others the so-called "diagonal method" in set theory, which are founded on such theses not accepted in the three- and infinite-valued systems of propositional logic. It would be interesting to inquire whether mathematical theorems based on the diagonal method could be demonstrated without propositional theses such as these [71].

(For those that don't speak Polish logic, the theses mentioned are $(\sim p \to p) \to p$, $(p \to \sim p) \to \sim p$, $(p \to q) \to ((p \to \sim p) \to q)$, $(p \to q \wedge \sim q) \to \sim p$ and $(p \to (q \leftrightarrow \sim q)) \to \sim p$.) I agree with Łukasiewicz in my estimation of the interest in inquiry concerning diagonal arguments in set theories without 'apagogic inference schemata' like contraction. Unfortunately, we won't get as far as that in this thesis. Chapters 11 and 12 contain a little exploration into mathematics without contraction, but the bulk of the work is still to come.

All of Łukasiewicz's logics are stronger than **CK**. It is easy to show that each axiom of **CK** is true in the infinite-valued logic $\mathbf{Ł}_\infty$. However, Łukasiewicz's logics themselves validate other strange inference schemata' which have escaped being suspect from time immemorial only by being uninteresting in their own right, and so, unexamined. However, when shown out in the open, there is little reason to take as true theses such as

$$(A \to B) \vee (B \to A) \quad \text{and} \quad \big((A \to B) \to B\big) \to \big((B \to A) \to A\big)$$

These fail in **CK** and so, in all of our favourite logics. Łukasiewicz's systems are useful as approximations to our logics, and not as models of valid inference in their own right. Their semantic structures — in terms of a linearly ordered set of truth values in $[0, 1]$ are overly simplistic, and they result in such odd theorems as those we just saw. We will show that the worlds semantics of our logics are more 'natural' and richer in terms of possible applications than Łukasiewicz's original proposals for many valued logics. For example, in Chapter 8 we shall see that logics in the vicinity of **CK** model vagueness better than $\mathbf{Ł}_\infty$.

Before going on to study semantic structures associated with our logics, we can prove a number of interesting facts about our systems, using the proof theory alone.

2.8 Deduction Theorems

A deduction theorem for a conditional helps us see the way that the conditional behaves. For example, in classical logic we can prove that

$$\Sigma \cup \{A\} \vdash B \text{ if and only if } \Sigma \vdash A \supset B$$

and then it is easy to show that properties of set union transfer to properties of the conditional. So, if $\Sigma \vdash A \supset (B \supset C)$ we must have $\Sigma \cup \{A, B\} \vdash C$ (applying the deduction theorem twice) and then $\Sigma \vdash B \supset (A \supset C)$ (applying the deduction theorem in the other order). The permutability of the conditional is inherited from the commutativity of set union. In just this way, the behaviour of intensional bunching affects the behaviour of the conditional by the general deduction theorem

Theorem 2.14 *In any of our favourite logics,* $A; X \Vdash B$ *if and only if* $X \Vdash A \to B$. *for each bunch* X *and formulae* A *and* B.

This result is a simple corollary of the definition of the natural deduction systems for our logics. However, the deduction theorem doesn't end there. We can prove similar results for weak validity. For this we need a new concept.

Definition 2.6 An *arbitrary fusion* of A is any way of fusing A together with itself (including the trivial fusion which is A itself). So, A is an arbitrary fusion of A, and if B and C are arbitrary fusions, then so is $B \circ C$.

For example, A^n for $n \geq 1$ is an arbitrary fusion of A, as is $A \circ ((A \circ A) \circ A)$, and so on.

Lemma 2.15 *If* $\Sigma \vdash A$ *and* C *is an arbitrary fusion of* A *then* $\Sigma \vdash C$ *too.*

Proof: By induction on the construction of C. If $C = A$ then the proof is trivial. Otherwise, suppose that the result holds for C_1 and C_2, so $\Sigma \vdash C_1$ and $\Sigma \vdash C_2$. Then $\vdash C_2 \to (C_1 \to C_1 \circ C_2)$ gives $\Sigma \vdash C_1 \circ C_2$ by two applications of *modus ponens*. This is sufficient for our result. ◁

Theorem 2.16 *In any of our favourite logics,* $\Sigma \cup \{A\} \vdash B$ *if and only if* $\Sigma \vdash C \to B$ *for some arbitrary fusion* C *of* $A \wedge t$.

Proof: If $\Sigma \vdash C \to B$ then clearly $\Sigma \cup \{A\} \vdash B$ because $\Sigma \cup \{A\} \vdash C$ by the previous lemma, and so *modus ponens* gives us the result.

The other direction is proved by induction on the length of the Hilbert proof from Σ, A to B. If it is of length 1, then B is either in Σ or an axiom, or it is A itself. In the first case, we have $\Sigma \vdash B$, giving $\Sigma \vdash t \to B$ and hence $\Sigma \vdash A \wedge t \to B$ In the second case, we have $\Sigma \vdash A \to B$, giving $\Sigma \vdash A \wedge t \to B$ as desired.

Now assume that the proof is of length m, and that the result holds for proofs of smaller length. Then B is either in $\Sigma \cup \{A\}$ or it follows from earlier statements in the proof by *modus ponens*. If the former, we're home. Suppose the latter. Then we have $\Sigma \cup \{A\} \vdash B'$ and $\Sigma \cup \{A\} \vdash B' \to B$ where these proofs are of shorter length than the proof of B. So the induction hypothesis applies to these proofs, and we have

$$\Sigma \vdash C_1 \to B' \qquad \Sigma \vdash C_2 \to (B' \to B)$$

for some arbitrary confusions C_1 and C_2 of $A \wedge t$. However, a little hacking with the Hilbert system shows that

$$C_1 \to B', \, C_2 \to (B' \to B) \vdash C_1 \circ C_2 \to B$$

(As $C_1 \to B' \vdash C_1 \circ C_2 \to B' \circ C_2$, and $C_2 \to (B' \to B) \vdash C_1 \circ C_2 \to B' \circ C_2$.) This means that $\Sigma \vdash C_1 \circ C_2 \to B$ which completes the induction step. ◁

Although this result is folklore for logics like **R** (where it holds as $\Sigma \cup \{A\} \vdash B$ if and only if $\Sigma \vdash A \wedge t \to B$ [86]) I have not seen it proved in this generality before.

This result simplifies for stronger logics. In any logics in which fusion is associative, such as **C**, **L** and **LI**, an arbitrary fusion of $A \wedge t$ is equivalent to $(A \wedge t)^n$ for some n (as we can associate to the right). In **CK** we can do better and eliminate the t, as $A \wedge t$ is equivalent to A, by Lemma 2.12.

This deduction theorem encodes the special properties of the contraction-free logics we consider. Each logic keeps tabs on how many times the antecedent was used in the proof. Logics up to **C** also keep tabs on *whether* an antecedent was used at all (and so, they need to strengthen the antecedent with a t in case the deduction of B from $\Sigma \cup \{A\}$ didn't use A). Logics below **C** keep a track of the order in which premises are used.

Note that the number of times that the premise A is used is not bounded by A or B. Any bound on this number is a function of the length of the proof. Calculating the length of a proof of B from some set is a notoriously difficult problem. *How* difficult it is to calculate the prospective fusion of $A \wedge t$, given Σ, A and B is unknown at the time of writing.

2.9 Metacompleteness

Finally, we will generalise Meyer's method of metavaluations [87] to get some proof theoretic control over our logics. This generalisation, due to Slaney [[]143] involves constructing a two-sorted metavaluation, which as we will show, generates the theorems of the logics. As Meyer originally defined metavaluations, these are intended to encode the way that connectives interact with theoremhood of our logics. We define two predicates: $\mathcal{T}$ for theorems, and $\mathcal{P}$ for 'possibilities' ($\mathcal{P}(A)$ just when $\sim A$ is not a theorem).

Firstly, we'll deal with **EW**, **C** and **CK**.

Definition 2.7 The *metavaluation pair* $\langle \mathcal{T}, \mathcal{P} \rangle$ of one of these logics is a pair of predicates on formulae, defined recursively as follows:

$\neg\mathcal{T}(p)$ for all atomic p	$\mathcal{P}(p)$ for all atomic p
$\mathcal{T}(t)$ $\quad$ $\neg\mathcal{T}(\bot)$	$\mathcal{P}(t)$ $\quad$ $\neg\mathcal{P}(\bot)$
$\mathcal{T}(A \wedge B)$ iff $\mathcal{T}(A)$ and $\mathcal{T}(B)$	$\mathcal{P}(A \wedge B)$ iff $\mathcal{P}(A)$ and $\mathcal{P}(B)$
$\mathcal{T}(A \vee B)$ iff $\mathcal{T}(A)$ or $\mathcal{T}(B)$	$\mathcal{P}(A \vee B)$ iff $\mathcal{P}(A)$ or $\mathcal{P}(B)$
$\mathcal{T}(\sim A)$ iff $\vdash \sim A$ and $\neg\mathcal{P}(A)$	$\mathcal{P}(\sim A)$ iff $\neg\mathcal{T}(A)$
$\mathcal{T}(A \to B)$ iff $\vdash A \to B$ and if $\mathcal{T}(A)$ then $\mathcal{T}(B)$ and if $\mathcal{P}(A)$ then $\mathcal{P}(B)$	$\mathcal{P}(A \to B)$ iff if $\mathcal{T}(A)$ then $\mathcal{P}(B)$
$\mathcal{T}(A \circ B)$ iff $\vdash A \circ B$ and $\mathcal{T}(A)$ and $\mathcal{T}(B)$	$\mathcal{P}(A \circ B)$ iff $\nvdash \sim(A \circ B)$ or $\mathcal{T}(A)$ and $(\mathcal{P}(B)$ or $\nvdash \sim B)$ or $\mathcal{P}(A)$ and $\mathcal{T}(B)$

We can then prove four results about the metavaluation pairs.

LEMMA 2.17 *If* $\mathcal{T}(A)$ *then* $\vdash A$.

LEMMA 2.18 *If* $\neg\mathcal{P}(A)$ *then* $\vdash \sim A$.

LEMMA 2.19 *If* $\mathcal{T}(A)$ *then* $\mathcal{P}(A)$.

LEMMA 2.20 *If* $\vdash A$ *then* $\mathcal{T}(A)$.

It follows from these that $\mathcal{T}(A)$ iff $\vdash A$. This will give us more control over provability in the logics, and we will be able to use this to learn more about the classes of theorems of these logics. Before this, however, we'll prove the lemmas. (The asterisks in the margin pick out places in the proof where restricted assertion features. These will be important later, but you can ignore them for now.)

Proof: (Of Lemma 2.17) This proceeds by a simple induction on the complexity of A. Clearly if A is p, t, F, $\sim B$, $B \to C$ or $B \circ C$ we have the result by construction. Otherwise, if A is $B \wedge C$, then the result follows by the property holding for B and C. Similarly for $A = B \vee C$. ◁

Proof: (Of Lemma 2.18) Again, an induction on the complexity of A. The results for p, t, F, and $B \circ C$ are immediate. For $A = B \wedge C$, if $\neg\mathcal{P}(B \wedge C)$ then either $\neg\mathcal{P}(B)$ or $\neg\mathcal{P}(C)$, so by hypothesis either $\vdash \sim B$ or $\vdash \sim C$. In either case, $\vdash \sim(B \wedge C)$. The case for disjunction is dual.

For $A = \sim B$, $\neg\mathcal{P}(\sim B)$ only if $\mathcal{T}(B)$, and so by the previous lemma, $\vdash B$. This gives $\vdash \sim\sim B$ by simple moves.

Similarly for $A = B \to C$, $\neg\mathcal{P}(B \to C)$ only if $\mathcal{T}(B)$ and $\neg\mathcal{P}(C)$. By hypothesis, $\vdash \sim C$ and by the previous lemma, $\vdash B$. However, in each of our logics under consideration, the $B \vdash (B \to C) \to C$, so we have $\vdash (B \to C) \to C$. By $\vdash \sim C$ we must have $\vdash \sim(B \to C)$ by *
contraposition. ◁

Proof: (Of Lemma 2.19) Yet again, we prove the claim by induction on the structure of A. The cases for p, t, and F are trivial. The conjunction and disjunction steps are immediate. For negation, suppose that $\mathcal{T}(\sim B)$ then by Lemma 2.17, $\vdash \sim B$ and by the consistency of the logics, $\nvdash B$. So, $\mathcal{P}(B)$.

If $A = B \to C$ and the result holds for C, suppose that $\mathcal{T}(B \to C)$. Then $\mathcal{P}(B \to C)$ *
because if $\mathcal{T}(B)$, we have $\mathcal{T}(C)$, and by the hypothesis holding for C, $\mathcal{P}(C)$ as desired.

Finally, if $A = B \circ C$ then if $\mathcal{T}(B \circ C)$ we have $\vdash B \circ C$ by construction, and by consistency, $\nvdash \sim(B \circ C)$, so $\mathcal{P}(B \circ C)$ as desired. ◁

Proof: (Of Lemma 2.20) Now the proof is by induction on the length of the proof in the logic under consideration. It suffices to show that each axiom is in the extension of $\mathcal{T}$, and that each rule is $\mathcal{T}$-preserving. Given this, we will have the result by a simple induction.

We'll tackle the rules first. Clearly *modus ponens* and conjunction introduction are $\mathcal{T}$-preserving. For restricted assertion assume $\mathcal{T}(A)$. We wish to show that $\mathcal{T}((A \to B) \to$ *
$B)$. By Lemma 2.17, $\vdash A$, so $\vdash (A \to B) \to B$ by restricted assertion. Assume $\mathcal{T}(A \to B)$. Then $\mathcal{T}(B)$ by $\mathcal{T}(A)$. Conversely, assume $\mathcal{P}(A \to B)$. Then if $\mathcal{P}(B)$ because $\mathcal{T}(A)$. So, we have $\mathcal{T}((A \to B) \to B)$ as desired.

For t-defining rules, assume that $\mathcal{T}(A)$. Then $\vdash t \to A$, and if $\mathcal{T}(t)$ then $\mathcal{T}(A)$, and if $\mathcal{P}(t)$ then $\mathcal{T}(A)$ which gives $\mathcal{P}(A)$ too, so $\mathcal{T}(t \to A)$ as desired. The converse is just as trivial.

For $\circ$-defining rules, assume that $\mathcal{T}(B \circ A \to C)$. We wish to show that $\mathcal{T}(A \to (B \to C))$. To this end, assume $\mathcal{T}(A)$. Then $\vdash A$ gives $\vdash B \to C$ (from $\vdash B \circ A \to C$). If $\mathcal{T}(B)$ then $\vdash B$ gives $\vdash B \circ A$ and so, $\mathcal{T}(B \circ A)$, which gives $\mathcal{T}(C)$. Also, $\mathcal{P}(B)$ gives $\mathcal{P}(B \circ A)$ and hence $\mathcal{P}(C)$ as desired. So, $\mathcal{T}(B \to C)$. Now assume $\mathcal{P}(A)$ to prove $\mathcal{P}(B \to C)$. For this, note that if $\mathcal{T}(B)$ we have $\mathcal{P}(B \circ A)$ which gives $\mathcal{P}(C)$ as desired.

For the converse of the rule, assume that $\mathcal{T}(A \to (B \to C))$, to prove $\mathcal{T}(B \circ A \to C)$. That $\vdash B \circ A \to C$ follows from the usual considerations. Assume that $\mathcal{T}(B \circ A)$. Then $\mathcal{T}(A)$, so $\mathcal{T}(B \to C)$, which from $\mathcal{T}(B)$ gives $\mathcal{T}(C)$. Now assume that $\mathcal{P}(B \circ A)$, to show that $\mathcal{P}(C)$. There are three possible cases for how $\mathcal{P}(B \circ A)$ could be made true. Firstly, $\nvdash {\sim}(B \circ A)$. Then it follows that $\nvdash C$ (as $\vdash B \circ A \to C$), and hence $\mathcal{P}(C)$ by. Secondly, $\mathcal{T}(B)$ and either $\mathcal{P}(A)$ or $\nvdash {\sim}A$. In either case, $\mathcal{P}(A)$ (by Lemma 2.18), so we have $\mathcal{P}(B \to C)$. Then $\mathcal{T}(B)$ gives $\mathcal{P}(C)$. Finally, $\mathcal{T}(A)$ and $\mathcal{P}(B)$. In this case, $\mathcal{T}(B \to C)$ so $\mathcal{P}(C)$ follows immediately.

The proof for the other $\circ$-defining rules (using backward implication, if present) is similar, and is left as an exercise.

For the transitivity rule, assume that $\mathcal{T}(A \to B)$ and $\mathcal{T}(C \to D)$, to show that $\mathcal{T}((B \to C) \to (A \to D))$. As usual, $\vdash (B \to C) \to (A \to D)$ is trivially provable. Assume $\mathcal{T}(B \to C)$. Then $\vdash A \to D$ by the usual moves. To show that $\mathcal{T}(A \to D)$, first assume $\mathcal{T}(A)$. Then $\mathcal{T}(D)$ follows from $\mathcal{T}(A \to B)$, $\mathcal{T}(B \to C)$ and $\mathcal{T}(C \to D)$. Secondly, assume $\mathcal{P}(A)$. As before, $\mathcal{T}(A \to B)$, $\mathcal{T}(B \to C)$ and $\mathcal{T}(C \to D)$ give $\mathcal{P}(D)$.

Now assume $\mathcal{P}(B \to C)$, to show that $\mathcal{P}(A \to D)$. If $\mathcal{T}(A)$ then $\mathcal{T}(B)$ (from $\mathcal{T}(A \to B)$) and $\mathcal{P}(B \to C)$ gives $\mathcal{P}(C)$. Then we have $\mathcal{P}(D)$ (from $\mathcal{T}(C \to D)$) as desired. This shows that transitivity is $\mathcal{T}$-preserving.

Axioms are quite similar, if tedious. Every axiom is a conditional, so we need only check that it is both $\mathcal{T}$- and $\mathcal{P}$-preserving. For conjunction elimination and disjunction introduction this is immediate from the definitions. For conjunction introduction, assume that $\mathcal{T}((A \to B) \wedge (A \to C))$. Then to show that $\mathcal{T}(A \to B \wedge C)$, assume that $\mathcal{T}(A)$. Then $\mathcal{T}(B)$ and $\mathcal{T}(C)$ give $\mathcal{T}(B \wedge C)$ as desired. Now assume that $\mathcal{P}(A)$. Then $\mathcal{P}(B)$ and $\mathcal{P}(C)$ give $\mathcal{P}(B \wedge C)$ as desired. Similarly, assume that $\mathcal{P}((A \to B) \wedge (A \to C))$ to show that $\mathcal{P}(A \to B \wedge C)$. Assume that $\mathcal{T}(A)$. Then $\mathcal{P}(B)$ and $\mathcal{P}(C)$ give $\mathcal{P}(B \wedge C)$ as desired. The case for disjunction elimination is dual.

Distribution follows immediately from distribution in the metalanguage. For assertion, and prefixing or suffixing (if present), the $\mathcal{T}$-preserving property follows from the same considerations that showed that the corresponding rules were $\mathcal{T}$-preserving. So, we need only show that they are also $\mathcal{P}$ preserving (in the right logics). So, for assertion, assume that $\mathcal{P}(A)$ to show that $\mathcal{P}((A \to B) \to B)$. For this, assume that $\mathcal{T}(A \to B)$. $\mathcal{P}(B)$ follows immediately. For prefixing, assume that $\mathcal{P}(A \to B)$ to show that $\mathcal{P}((B \to C) \to (A \to C))$. For this, assume that $\mathcal{T}(B \to C)$ to show that $\mathcal{P}(A \to C)$, so assume $\mathcal{T}(A)$. Then $\mathcal{P}(B)$ from $\mathcal{P}(A \to B)$, so $\mathcal{P}(C)$ as desired. Suffixing is similar.

For contraposition, assume that $\mathcal{T}(A \to {\sim}B)$ to show that $\mathcal{T}(B \to {\sim}A)$. If $\mathcal{T}(B)$ then $\neg\mathcal{P}({\sim}B)$, and so, $\neg\mathcal{P}(A)$, giving $\mathcal{T}({\sim}A)$. If $\mathcal{P}(B)$ then $\neg\mathcal{T}({\sim}B)$ so $\neg\mathcal{T}(A)$, giving

$\mathcal{P}({\sim}A)$. So, $\mathcal{T}(B \to {\sim}A)$. Now assume that $\mathcal{P}(A \to {\sim}B)$ to show that $\mathcal{P}(B \to {\sim}A)$. If *
$\mathcal{T}(B)$ then $\neg\mathcal{P}({\sim}B)$ so $\neg\mathcal{T}(A)$, giving $\mathcal{P}({\sim}A)$ as desired.

For double negation, assume that $\mathcal{T}({\sim}{\sim}A)$. Then $\neg\mathcal{P}({\sim}A)$ and so, $\mathcal{T}(A)$ as desired. If $\mathcal{P}({\sim}{\sim}A)$ then $\neg\mathcal{T}({\sim}A)$, so either $\nvdash {\sim}A$ or $\mathcal{P}(A)$. In the second case we're done. In the first, $\nvdash {\sim}A$ gives $\neg\mathcal{T}({\sim}A)$ and hence, $\mathcal{P}(A)$ as desired.

For fusion axioms (if present) the $\mathcal{T}$-preservation part has already been proved. Assume that $\mathcal{P}((B \circ A) \to C)$ in order to show that $\mathcal{P}(A \to (B \to C))$. If $\mathcal{T}(A)$ then if $\mathcal{T}(B)$ we have $\mathcal{T}(B \circ A)$, giving $\mathcal{P}(C)$ as desired. Conversely, assume that $\mathcal{P}(A \to (B \to C))$
in order to show that $\mathcal{P}((B \circ A) \to C)$. Here, if $\mathcal{T}(B \circ A)$ then $\mathcal{T}(A)$ and $\mathcal{T}(B)$ give $\mathcal{P}(C)$ *
as desired. The case for the converse implication axioms are left as an exercise.

For t-axioms we need only check $\mathcal{P}$-preservation. If $\mathcal{P}(A)$ then if $\mathcal{T}(t)$ we have $\mathcal{P}(A)$, *
so $\mathcal{P}(t \to A)$. Conversely, if $\mathcal{P}(t \to A)$, then $\mathcal{T}(t)$ gives $\mathcal{P}(A)$ as desired. ◁

As Slaney has said [143] there ought to be a smoother way of doing this, but we don't know it.

Recall that restricted assertion features only in **EW**, **LI**$^+$, and stronger logics. To modify this result to work with logics without restricted assertion, we need to modify the way that $\mathcal{P}$ works for conditionals. Demand that $\mathcal{P}(A \to B)$ *always*. Then the proof goes through without appeal to restricted assertion. The reader is invited to reread the proofs, modifying the places indicated by asterisks. With the modification to the negated conditional fact, the proof works for **DW** and **TW**.

It's even easier to extend the logics **L**$^+$ and **LI**$^+$. For this we cut out the conditions for negation, the clauses for $\mathcal{P}$ and the $\mathcal{P}$ conditions in the clause for $\mathcal{T}(A \to B)$. Then the proof of Lemma 2.17 and Lemma 2.20 go through as is, without any need for the intervening lemmas. The details are left to the enthusiastic.

Given these results we have a pleasing theorem for all of our favourite logics.

Theorem 2.21 $\mathcal{T}(A)$ *iff* $\vdash A$, *and* $\mathcal{P}(A)$ *iff* $\nvdash {\sim}A$.

This has some nice corollaries.

Corollary *In each of our favourite logics,* $\vdash A \vee B$ *iff* $\vdash A$ *or* $\vdash B$. *That is, each of our logics are prime.*

Corollary *In our favourite logics that contain restricted assertion,* $\vdash {\sim}(A \to B)$ *iff* $\vdash A$ *and* $\vdash {\sim}B$. *So, a conditional is refutable in these logics just when the logic has an explicit refutation.*

Corollary *In our favourite logics without restricted assertion,* $\nvdash {\sim}(A \to B)$ *for each* A *and* B.

This is one consideration that indicates that logics with restricted assertion are better than those without. If there is an explicit counterexample to a conditional $A \to B$ as a matter of logic — that is, if $\vdash A$ and $\vdash {\sim}B$ — it is plausible that the conditional is false, as a matter of logic. So, it is plausible that $(A \to A) \to {\sim}(B \to B)$ is false, as a matter of logic. But we have just shown that in systems such as **TW** and below, $\nvdash {\sim}\big((A \to A) \to {\sim}(B \to B)\big)$. Logics like these are weak with respect to false conditionals.

2.10 Validities

The two concepts of validity discussed in this chapter are not the only ones. Given the choice between strong validity, for which *modus ponens* is not valid (because some bunches are not closed under *modus ponens*) and weak validity, in which the premises are hypothesised as a matter of *logic* (making $\Box A$ follow from A) it makes sense to search about for more options. After all, neither of these particularly match the way we tend to reason. When reasoning from a bunch of premises, at least sometimes, we suppose that the *world* is like that, and then apply our deductive machinery to see what follows. Let's see how we can generalise our kinds of validity to pick out a 'world-like' notion of deduction suitable for our needs. Firstly, note our definition of $\Sigma \vdash A$.

$$\text{If } 0 \Vdash B \text{ for each } B \in \Sigma \text{ then } 0 \Vdash A.$$

Now $\Sigma \Vdash A$ could be defined similarly (where Σ is still a set of assumptions). Let it mean this:

$$\text{If } X \Vdash B \text{ for each } B \in \Sigma \text{ then } X \Vdash A, \text{ for } \textit{every} \text{ bunch } X.$$

Then $\Sigma \Vdash A$ if and only if the extensional bunch of each element of Σ supports A, in the old sense. This definition brings out the obvious connection between strong and weak validity. Strong validity amounts to preservation at every bunch. We simply assume that some bunch supports all our premises (which amounts to assuming a bunch that supports its extensional conjunction) and we then see whether that bunch supports the conclusion. Weak validity is less ambitious. Instead of considering an arbitrary bunch supporting all of the elements of Σ, it considers only the logic bunch 0.

There is an obvious generalisation. We could define *world* validity, $\Sigma \vdash_W A$ as follows:

$$\text{If } X \Vdash B \text{ for each } B \in \Sigma \text{ then } X \Vdash A, \text{ for every } \textit{world} \text{ bunch } X.$$

provided that we have a class of world bunches to choose from. This would allow us the freedom to specify more closely what the world is like, and to get more interesting valid arguments. For example, it is plausible to suppose that the world is closed under *modus ponens*. That is, if $W \Vdash A$ and $W \Vdash A \to B$ then $W \Vdash B$ where W is a world bunch. Then we simply guarantee that world bunches have this property. And that is easy — we require that if W is a world, then $W \Longleftarrow W; W$. Other simple properties for worlds might be that $W \Vdash t$ (worlds agree with logic) and perhaps, $W \Vdash A \vee {\sim} A$ (worlds are complete). Then we will have world validities like $A, A \to B \vdash_W B$, as you would expect, without such oddities as $A \vdash_W t \to A$. This kind of validity is a way to allow limited forms of *modus ponens* and contraction, without making the *logic* itself collapse into admitting contraction.

This conception of world validity merits a good deal of further study. (The questions arise: How to provide a Hilbert-style axiomatisation for various notions of $\vdash_W$. Presumably it will be similar to the Hilbert proofs in modal logics in which certain rules (such as necessity introduction) only apply to axioms and their consequences. Parallel to this question, the issue arises: what are ‘natural’ appropriate conditions on world bunches? In logics like ours, $\Vdash$ is decidable (Chapter 7) but $\vdash$ is not. What about $\vdash_W$? Other questions will arise when we discuss world validity in connection with the semantic structures associated with our logics.) However, this is not the place to answer questions like this, because the concept became clear only in the very late stages of preparation of the thesis. We will have to make do with small, periodic excursions to show how world validity can be defined in the semantic structures of our logics, and how these can be interpreted along with the semantics. Further work requires another time, and another place.

2.11 Notes

[1]We will also add other connectives such as $\leftarrow$, $\top$ or $\bot$ as the need arises. It is to be hoped that the reader can cope with this addition to the language, or ‘progressive revelation’ of was already there.

[2]The names are standard in the literature. They are named for their obvious structural similarity to the combinators of the same name [94].

[3]One way to see the interderivability at the level of theorems is this: If $\sim$ satisfies axiom contraposition (in that $0 \Vdash (A \to B) \leftrightarrow ({\sim}B \to {\sim}A)$) then from an instance of suffixing, $(A \to B) \to \big((B \to C) \to (A \to C)\big)$ you can deduce $({\sim}B \to {\sim}A) \to \big(({\sim}C \to {\sim}B) \to ({\sim}C \to {\sim}A)\big)$, which is an instance of prefixing. It follows that if we have every instance of suffixing, we have every instance of prefixing of negations. If every proposition is equivalent to a negation (for example, if we have double negation) then we have all instances of prefixing. The same holds for the other direction.

[4]See Section 2.10 for an explanation of how these concepts of validity relate, and what other notions of validity lurk in between weak and strong validity.

[5]But in **C+ExMid** the paradoxes emerge *despite* the absence of contraction in general. The limited form of contraction given by **ExMid** is enough to push through the paradoxical derivations [147].

Chapter 3
Propositional Structures

Propositions ...
are divided into theorems
and problems.
— EUCLID *The Elements* [39]

The proof theory of the previous chapter is very useful. It gives us a handle on what follows from what in each of our logics. But it has one practical shortcoming. The proof theory gives us no simple way to show that an argument is invalid. For example the formula something like

$$((A \to B) \to B) \to A \tag{α}$$

is classically unprovable. How would you demonstrate this using proof theory alone? You can start by showing that classically $\vdash C \to D \vee {\sim} D$ for each D and that therefore $\vdash (A \to D \vee {\sim} D) \to D \vee {\sim} D$, which by *modus ponens* would give $\vdash A$ if (α) were provable. So, we can show that if (α) is provable, anything is. But how can we show that classical logic is consistent, using proof theory alone? There is no simple way to do this. The metacompleteness argument of the previous chapter gives us some kind of control over provability in our logics, but that method isn't simple, and it doesn't give us general results about arbitrary formulae.

However, there is a simple way to show that a formula is unprovable in the classical propositional calculus, by *truth tables*. A simple assignment of truth values to terms like this:

$$\begin{array}{ccccccc} ((A & \to & B) & \to & B) & \to & A \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 \end{array}$$

shows that (α) is false when A is false and B is true, under the usual evaluation conditions for classical truth tables. Once we are assured that the axioms of classical propositional logic are true under any evaluation into truth tables, and that *modus ponens* preserves truth under evaluation, then we know that (α) is not a theorem. For otherwise, it would have no counterexample in truth tables. We will generalise this method to give us a way to show non-theoremhood for the logics we're studying.

In this chapter we will generalise truth tables by defining classes of *propositional structures* for our logics. Weak validity amounts to truth preservation in these structures and there will be a lot in common with the traditional story from classical logic.

3.1 Definitions

Consider propositions; whatever they are, they are ordered by the relation $\leq$ of valid deduction. That is, $a \leq b$ just when b follows from a, when a gives at least as much information as b, or any other equivalent condition. Clearly $\leq$ is transitive and reflexive. Furthermore, propositions simply *are* their inferential force, so it is sensible to assume that if $a \leq b$ and $b \leq a$, then $a = b$: $\leq$ is antisymmetric, and hence, we have a partial order on propositions.

As we saw in our discussion at the start of the previous chapter, there are two ways taking propositions together. The first, extensional conjunction, is the greatest lower bound of $\leq$. The proposition $a \cap b$ is the weakest proposition that entails both a and b. For this to work, we need just ensure that $\leq$ is a semilattice ordering with greatest lower bound $\cap$. The second way of taking together, intensional conjunction, (written '$\cdot$') is harder to pin down. At the very least, it is monotonic with respect to $\leq$, in that if $a \leq b$ and $c \leq d$ then $a \cdot c \leq b \cdot d$ — this readily follows from the fact that it is a form of taking together. Taking more than a together with more than b results in more than just taking a and b together. This form of taking together also has a right identity e — the 'logic' proposition — so $a \cdot e = a$ for each a. A structure satisfying these minimal conditions is a *bare* **DW** *propositional structure*.

We may be interested in more connectives that conjunction and fusion. If so, we add more connectives to a bare **DW** propositional structure. For disjunction we add the condition that $\leq$ be a distributive lattice with least upper bound $\cup$. In addition, we ask that $\cup$ respect $\cdot$, so $(a \cup b) \cdot c = (a \cdot c) \cup (b \cdot c)$ and $a \cdot (b \cup c) = (a \cdot b) \cup (a \cdot c)$.

For implication we add a residual $\Rightarrow$ for fusion. We make $\Rightarrow$ a binary operation satisfying the residuation condition

$$b \leq a \Rightarrow c \text{ if and only if } ab \leq c$$

And for negation we add a unary operation $\overline{}$ satisfying the de Morgan identities and the double negation postulate

$$\overline{a \cup b} = \overline{a} \cap \overline{b} \qquad \overline{a \cap b} = \overline{a} \cup \overline{b} \qquad a = \overline{\overline{a}}$$

as well as the contraposition law

$$a \cdot b \leq \overline{c} \text{ if and only if } c \cdot b \leq \overline{a}$$

(Note that these definitions straightforwardly encode the insights of the natural deduction system of Chapter 2 into the context of propositions: we identify provably equivalent statements, and consider the algebraic structure induced by the logical operations.)

Given all of this, we have a **DW** propositional structure.

Definition 3.1 $\mathfrak{P} = \langle P; \leq, \cdot, e \rangle$ is a *bare* **DW** *propositional structure* (or a bare **DW**ps) just when

- $\langle P; \leq \rangle$ is a semilattice.
- The binary operator '$\cdot$' is monotone with respect to $\leq$, so $a \leq b$ and $c \leq d$ only if $a \cdot c \leq b \cdot d$.
- The constant e is a right identity for fusion: $a \cdot e = a$.

If $\mathfrak{P}$ is a bare **DW**ps and $\leq$ is a lattice order with least upper bound $\cup$, which also satisfies $(a \cup b) \cdot c = (a \cdot c) \cup (b \cdot c)$ and $a \cdot (b \cup c) = (a \cdot b) \cup (a \cdot c)$ and $a \cap (b \cup c) = (a \cap b) \cup (a \cap c)$ then $\mathfrak{P}$ is a bare **DW**ps *with disjunction*.

If $\mathfrak{P}$ is a bare **DW**ps and $\overline{}$ a unary operation such that $\overline{\overline{a}} = a$, $\overline{a \cup b} = \overline{a} \cap \overline{b}$, $\overline{a \cap b} = \overline{a} \cup \overline{b}$ and $ab \leq \overline{c}$ if and only if $cb \leq \overline{a}$, then the resulting structure is a bare **DW**ps *with negation*.

If $\mathfrak{P}$ is a bare **DW**ps and $\Rightarrow$ a binary operation that is a right residual for fusion (so $a \cdot b \leq c$ iff $b \leq a \Rightarrow c$), then the resulting structure is a bare **DW**ps *with a right residual*.

If $\mathfrak{P}$ is a bare **DW**ps and $\Leftarrow$ a binary operation that is a left residual for fusion (so $a \cdot b \leq c$ iff $a \leq c \Leftarrow b$), then the resulting structure is a bare **DW**ps *with a left residual*.

If $\mathfrak{P}$ is a bare **DW**ps and F a constant such that $F \leq a$ for each a, then the resulting structure is a bare **DW**ps *with a bottom*.

Finally, a $\mathbf{DW}^+$ps is a bare **DW**ps with disjunction and a right residual. A **DW**ps is $\mathbf{DW}^+$ps with negation.

If the propositional structure satisfies more conditions, it will model stronger logics. We define a few conditions and their corresponding logics here:

Definition 3.2

- A **TW**ps is a **DW**ps in which $a \cdot (b \cdot c) \leq (a \cdot b) \cdot c$ and $a \cdot (b \cdot c) \leq (a \cdot c) \cdot b$. (And a bare **TW**ps is a bare **DW**ps in which $a \cdot (b \cdot c) \leq (a \cdot b) \cdot c$ and $a \cdot (b \cdot c) \leq (a \cdot c) \cdot b$, and so on.)
- An **EW**ps is a **TW**ps in which $a \leq 0 \cdot a$.
- A **C**ps is a **DW**ps in which $b \cdot (a \cdot c) \leq a \cdot (b \cdot c)$.
- A **CK**ps is a **C**ps in which $a \cdot b \leq b$.
- An $\mathbf{L}^+$ps is a $\mathbf{DW}^+$ps in which $a \cdot (b \cdot c) = (a \cdot b) \cdot c$, together with a left residual.
- An $\mathbf{LI}^+$ps is an $\mathbf{L}^+$ps in which $a = 0 \cdot a$.

The following results are easy to show

LEMMA 3.1

- *A structure is a* **TW***ps if and only if it is a* **DW***ps in which either* $a \cdot (b \cdot c) \leq (a \cdot b) \cdot c$ *or* $a \cdot (b \cdot c) \leq (a \cdot c) \cdot b$.
- *A structure is a* **C***ps if and only if it is a* **TW***ps in which* $a \cdot b = b \cdot a$.
- *A structure is a* **CK***ps if and only if it is a* **C***ps in which* $a \leq e$ *for each* a.

To tidy up notation, we call a **DW**ps that models the connectives in the set C (where $\wedge, \circ, t \in C$) a $\mathbf{DW}_C$ps. So, a bare **DW**ps is also a $\mathbf{DW}_{\wedge\circ t}$ps, while a **DW**ps is a $\mathbf{DW}_{\wedge\vee\circ\to\sim t}$ps and a **L**ps is a $\mathbf{L}_{\wedge\vee\circ\to\leftarrow t}$ps, and so on.

Structures like these were introduced by Dunn [3, 27, 33] (who introduced **R** propositional structures (de Morgan monoids) to model the relevant logic **R**) and then generalised by Meyer and Routley [94] who used similar structures to model a very large range of systems.

3.2 Soundness and Completeness

Given propositional structures it is a simple matter to model valid inference in the propositional fragments of our logics. (We leave discussion of the quantifiers to the next chapter. For the moment assume we're working in the propositional fragments of our logics.)

Definition 3.3 Given a propositional structure $\mathfrak{P} = \langle P; \leq, \cdot, e \rangle$ an *interpretation* of a propositional language $\mathcal{L}$ in $\mathfrak{P}$ is a map $I : \mathcal{L} \to P$ that satisfies the following recursive conditions:

$$I(A \wedge B) = I(A) \cap I(B) \qquad I(A \circ B) = I(A) \cdot I(B) \qquad I(t) = e$$

and for each of the extra connectives that are given in the propositional structure, the appropriate conditions from among

$$I(A \vee B) = I(A) \cup I(B) \qquad I({\sim} A) = \overline{I(A)} \qquad I(\bot) = F$$

$$I(A \to B) = I(A) \Rightarrow I(B) \qquad I(A \leftarrow B) = I(B) \Leftarrow I(B)$$

also hold. Then a sentence A is *true under* I just when $e \leq I(A)$. A sentence is *true in the structure* $\mathfrak{P}$ just when it is true under all interpretations in that structure. A sentence is *true in a class of structures* just when it is true in all structures in that class.

Similarly, a deduction from Σ to A is *weakly valid in the structure* $\mathfrak{P}$ just when for each I under which every element of Σ is true, A is true under I as well. The deduction from Σ to A is *weakly valid in a class of structures* just when it is type one valid in all structures in that class.

Strong validity is also modellable. Extend I to map bunches and by adding the conditions

$$I(X, Y) = I(X) \cap I(Y) \qquad I(X; Y) = I(X) \cdot I(Y)$$

A deduction $X \Vdash A$ is *strongly valid under* I when $I(X) \leq I(A)$. It is *strongly valid* in $\mathfrak{P}$ (in a class of structures) just when it is strongly valid under each interpretation I in $\mathfrak{P}$ (in each $\mathfrak{P}$ in the class).

THEOREM 3.2 **Soundness and Completeness**. *A deduction from Σ to A is weakly valid in the class of all* $\mathbf{DW}_C$ *propositional structures if and only if $\Sigma \vdash A$ in* $\mathbf{DW}_C$*, where C is a connective set including $\wedge$, $\circ$ and t. Similarly for* **TW**, **EW**, $\mathbf{L}^+$, $\mathbf{LI}^+$, **C** *and* **CK** *in the place of* **DW**.

Proof: It is tedious but simple to show that if $\Sigma \vdash A$ in $\mathbf{DW}_C$ then the deduction from Σ to A is valid in in the class of all $\mathbf{DW}_C$ propositional structures. Take a $\mathbf{DW}_C$ propositional structure an interpretation I on that structure such that each element of Σ is true under I. We will show that A is true under I too. Consider a natural deduction proof of A under the hypothesis of Σ. We will show that each sequent mentioned in this proof is strongly valid under I. Once this is done, the conclusion (which states that $0 \Vdash A$) shows that A is true under I.

Clearly each axiom and hypothesis in the proof is valid under I. We need just show that each natural deduction rule preserves validity. The structural rules preserve validity, as they mirror identities in the propositional structures. For the logical rules we will sketch the proofs for some, and leave the rest for the reader. For $\wedge$E, note that if $I(X) \leq I(A \wedge B)$ then $I(A \wedge B) = I(A) \cap I(B) \leq I(A)$ gives the result that $I(X) \leq I(A)$ as desired. For CP, if $I(A; X) \leq I(B)$ then $I(A) \circ I(X) \leq I(B)$ so $I(X) \leq I(A) \Rightarrow I(B) =$

$I(A \to B)$ as desired. Finally, for MP, if $I(X) \leq I(A)$ and $I(Y) \leq I(A \to B) = I(A) \Rightarrow I(B)$ then $I(X;Y) = I(X) \cdot I(Y) \leq I(A) \cdot I(A \to B) \leq I(B)$ as desired. (As $a \cdot (a \Rightarrow b) \leq b$ by residuating $a \Rightarrow b \leq a \Rightarrow b$.) The rest are left for the enthusiastic reader.

For the extensions of **DW** it suffices to note that the structural rules mirror exactly the extending conditions for the propositional structures.

For the other direction we wish to show that if $\Sigma \nvdash A$ in one of our logics, there is an appropriate propositional structure and interpretation in which Σ comes out as true, but A does not. For this, take the propositions in the structure to be equivalence classes of formulae, where $A \sim B$ just when $\Sigma \vdash A \leftrightarrow B$. Then we set $[A] \leq [B]$ just when $\Sigma \vdash A \to B$ (this is well defined, as if $\Sigma \vdash A_1 \leftrightarrow A_2$, $\Sigma \vdash B_1 \leftrightarrow B_2$ and $\Sigma \vdash A_1 \to B_1$ then $\Sigma \vdash A_2 \to B_2$ by a simple extension of Theorem 2.9). This is a semilattice ordered by $\cap$, where $[A] \cap [B] = [A \wedge B]$, and if disjunction is present, then $\cup$ where $[A] \cup [B] = [A \vee B]$ makes this a distributive lattice. The proofs are easy.

Then define the other operations on classes in the obvious way: $[A] \cdot [B] = [A \circ B]$ and $e = [t]$, and if present, $[A] \Rightarrow [B] = [A \to B]$, $[A] \Leftarrow [B] = [A \leftarrow B]$, $\overline{[A]} = [\sim A]$ and $F = [\bot]$. It is not difficult to show that the required conditions on a propositional structure are satisfied by this construction, if the relevant connectives are present. For example, we will show that $\big([A] \cup [B]\big) \cdot [C] = \big([A] \cdot [C]\big) \cup \big([B] \cdot [C]\big)$. For one direction, note that $\vdash C \to \big(A \to (A \circ C) \vee (B \circ C)\big)$ and similarly, $\vdash C \to \big(B \to (A \circ C) \vee (B \circ C)\big)$ so $\vdash C \to \big(A \vee B \to (A \circ C) \vee (B \circ C)\big)$ giving $\vdash (A \vee B) \circ C \to (A \circ C) \vee (B \circ C)$ as desired. Conversely, $\vdash A \to A \vee B$ gives $\vdash A \circ C \to (A \vee B) \circ C$. Similarly $\vdash B \circ C \to (A \vee B) \circ C$, so $(A \circ C) \vee (B \circ C) \to (A \vee B) \circ C$.

It is also needed to show that if the logic is stronger than **DW** then the generated propositional structure is of the right kind. Again, we do the case for **C** and leave the rest. Now we wish to show that $[B] \cdot \big([A] \cdot [C]\big) \leq [A] \cdot \big([B] \cdot [C]\big)$. Firstly $\vdash C \to \big(B \to (A \to A \circ (B \circ C))\big)$ and prefixing C onto the **C**-theorem $\vdash \big(B \to (A \to A \circ (B \circ C))\big) \to \big(A \to (B \to A \circ (B \circ C))\big)$ gives $\vdash C \to \big(A \to (B \to A \circ (B \circ C))\big)$ which when unresiduated gives $\vdash B \circ (A \circ C) \to A \circ (B \circ C)$ as desired.

The interpretation on this structure is simple: $I(A) = [A]$ for each A. It trivially satisfies the inductive definitions. In only remains to show that under I, Σ is true, and A is not. But this is trivial, as if $B \in \Sigma$ then $\Sigma \vdash B$ and hence $\Sigma \vdash t \to B$, which means that $e \leq [B]$. But $e \not\leq [A]$ as $\Sigma \nvdash A$. So this is the countermodel we require. ◁

We will call the structure we have defined a Σ *formula algebra*. This structure gives the next result too.

THEOREM 3.3 **Soundness and Completeness 2**. *A deduction from* $X \Vdash A$ *is strongly valid in the class of all* $\mathbf{DW}_C$ *propositional structures if and only if* $X \Vdash A$ *in* $\mathbf{DW}_C$, *where* C *is a connective set including* $\wedge$, $\circ$ *and* t. *Similarly for* **TW**, **EW**, $\mathbf{L}^+$, $\mathbf{LI}^+$, **C** *and* **CK** *in the place of* **DW**.

Proof: Simple. As before, if $X \Vdash A$ then we can show that $I(X) \leq I(A)$ for each I by examining the proof that $X \Vdash A$. The axioms are all valid, and each step preserves validity under I, in the relevant structures (for the structural rules for the stronger logics). So, the conclusion is valid.

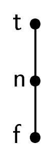
For the converse, if $X \nvDash A$ we use the $\varnothing$ formula algebra, (taking Σ to be empty) and then note that as $X \nvDash A$, $I(X) \not\leq I(A)$ as desired. $\triangleleft$

These soundness and completeness proofs are rather trivial. This is because the semantics is almost 'read off' the syntax. Later on we shall see semantic structures which are less closely related to the syntax, and which give us more information about the logics. However, we shall see that propositional structures still give us helpful information about our logics.

3.3 Example Structures

Some of the oldest contraction-free logics are defined in terms of propositional structures. These are due to Jan Łukasiewicz [70].

Łukasiewicz's Logics Łukasiewicz's three valued logic $\mathbf{Ł}_3$ is often useful in the study of contraction-free logics. The three element algebra is a **CK**ps.



$\Rightarrow$	t	n	f
*t	t	n	f
n	t	t	n
f	t	t	t

	$-$
*t	f
n	n
f	t

The Hasse diagram on the left is read in the usual way: $f < n < t$. Implication is given there, and $a \cdot b$ is defined as $\overline{a \Rightarrow \overline{b}}$. The true element t is the fusion identity e. It is tedious but straightforward to check that this is, in fact, a **CK**ps — you need just check that the operations on the structure satisfy the **CK** conditions.

We can use this propositional structure to check that some things we thought wouldn't be **CK** theorems (or valid rules) really aren't theorems (or valid rules). For example, *pseudo modus ponens* fails, because $n \cap (n \Rightarrow f) \Rightarrow f = n \cap n \Rightarrow f = n$.

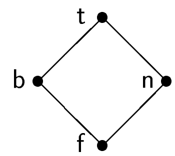
Similarly $A \to (A \to B) \not\vdash A \to B$ because $n \Rightarrow (n \Rightarrow f) = n \Rightarrow n = t$ but $n \Rightarrow f = n$. So, this structure invalidates both *pseudo modus ponens* and the contraction rule. We have finally shown that each of our favourite logics is free of contraction.

Łukasiewicz's other systems are all sub-systems of $\mathbf{Ł}_3$, in that they have fewer theorems. The most important of these is the infinitely valued logic $\mathbf{Ł}_\infty$. Here the domain is the interval $[0, 1]$, and the operations are given by setting

$$
\begin{aligned}
a \Rightarrow b &= \min(1, 1 - a + b) \\
\overline{a} &= 1 - a = a \Rightarrow 0 \\
a \cup b &= \max(a, b) = (a \Rightarrow b) \Rightarrow b \\
a \cap b &= \min(a, b) = \overline{\overline{a} \cup \overline{b}} \\
a \cdot b &= \max(0, a + b - 1) = \overline{a \Rightarrow \overline{b}}
\end{aligned}
$$

with 1 being the true element e.

BN4 Another interesting structure gives the logic **BN4**. Belnap [15], Meyer, Giambrone and Brady [91] and Slaney [152] all extol its virtues. The structure is simple.



$\Rightarrow$	t	b	n	f
*t	t	f	n	f
*b	t	b	n	f
n	t	n	t	n
f	t	t	t	t

	$-$
*t	f
*b	b
n	n
f	t

Fusion is still defined by setting $a \cdot b = \overline{a \Rightarrow \overline{b}}$, and t is e. It is a pleasing exercise to show that **BN4** properly contains **C**. That it contains **C** is simply a matter of axiom chopping — which we unashamedly leave to the reader. To show this containment is proper, we give the reader a hint. Consider the sentence:

$$(p \to q) \vee (q \to r) \vee (q \leftrightarrow \sim q)$$

It is not a theorem of **C** (as none of its disjuncts are — recall the first corollary to Theorem 2.11) but it comes out designated in **BN4** every time. (Due to a limited number of places to put 'q.' If $(p \to q) \vee (q \to r)$ is to fail, q must be either of b or n — and it then makes $q \leftrightarrow \sim q$ true.) **BN4** is weaker than **CK**, because weakening axiom $p \to (q \to p)$ fails — send p to t and q to b. Contraction fails in **BN4** because it contains $\mathbf{Ł}_3$ as a substructure.

Relation Algebras Relation algebras provide another independent source of propositional structures. These date to before the turn of the century, and they provide an elegant generalisation of boolean algebras. Luminaries such as de Morgan, Peirce, Russell and Schröder all wrote about the calculus [98, 109, 135, 140]. However, work on relation algebra went into a decline as the new century progressed and first-order logic gained popularity. It was the ground breaking work of Tarski, Jónsson, Lyndon and others that brought relation algebras back into view [59, 74, 158, 159]. Their relationship with logic like ours is explained by Dunn [35].

These algebras are a generalisation of boolean algebras. Take a class of objects and a collection of *binary relations* on this class. There are a number of ways to form new relations from old: the boolean connectives are some, but there are also others. Pairs of relations can be *composed*: a and b are related by the composition of ρ and σ (written $\rho;\sigma$) just when for some c, $a\rho c$ and $c\sigma b$. So, `parent-of;parent-of` is the relation `grandparent-of`. Composition is associative, but not symmetric or idempotent. The *identity* relation 1' satisfies $1';\rho = \rho = \rho;1'$. Any relation ρ has an *inverse*, written $\rho^{\smile}$, which satisfies $\rho^{\smile\smile} = \rho$ and a number of other identities.

We can generalise this picture to form an abstract *relation algebra*, as a 7-tuple $\langle R, \cup, -, 1, ;, {}^{\smile}, 1' \rangle$, such that $\langle R, \cup, -, 1 \rangle$ is a boolean algebra with top element 1 (the full relation) ; (composition) is an associative binary operation on R with identity 1' (the identity relation) and $^{\smile}$ (inverse) is a unary operation satisfying:

$$(x;y)^{\smile} = y^{\smile};x^{\smile} \qquad x^{\smile\smile} = x \qquad (x \cup y)^{\smile} = x^{\smile} \cup y^{\smile}$$

In addition, composition distributes over disjunction:

$$x;(y \cup z) = (x;y) \cup (x;z) \qquad (x \cup y);z = (x;z) \cup (y;z)$$

It is easy to see that each of these are true under the intended interpretation of relation algebras. The variety of all algebras satisfying these conditions is called **RA**.

A significant result in the theory of **RA** is that composition is *left and right residuated*. Specifically, if we set:

$$x \Rightarrow y = -(x^{\smile};-y) \qquad y \Leftarrow x = -(-y;x^{\smile})$$

we have $x;y \leq z$ if and only if $y \leq x \Rightarrow z$ if and only if $y \leq z \Leftarrow x$ (where $a \leq b$ is defined as $a \cup b = b$).

It would be pleasing to show that every relation algebra is isomorphic to a *concrete* relation algebra (a collection of relations, closed under each of the operations, given their standard interpretation) but this is not the case, as Lyndon proved [74].

Despite this, the variety **RA** of relation algebras are interesting in themselves. Their interest to us lies in the fact that they are closely related to $\mathbf{LI}^+$. In fact, it is simple to show the $\mathbf{LI}^+$ propositional structures are exactly these $\mathbf{RA}^+$ algebras, where we take an $\mathbf{RA}^+$ algebra to be a structure

$$\mathfrak{R} = \langle R, \cup, \cap, 1, ;, 1', \Leftarrow, \Rightarrow \rangle$$

that satisfies each of the axioms of **RA** in its vocabulary. That is, we keep the positive (boolean negation and converse-free) part of the algebra, and admit the two residuals defined by their residuation clauses. So, relation algebras provide a large class of models for logics like $\mathbf{LI}^+$.

Is there any negation that can extend this to model one of our logics containing negation? No. There are two natural contenders. The first is boolean negation. This will be too strong, in that it will validate noxious theses such as $A \wedge -A \to B$ and $A \to B \vee -B$, which are not provable in any of our favourite logics. The other contender is $^{-}$ given by setting $\overline{a} = -(a^{\smile}) = (-a)^{\smile}$. This comes closer to negation as modelled in our favourite logics: It doesn't validate any noxious theses given by boolean negation. However, both these candidates for negation fail with respect to contraposition. Here is an example. Let $\mathfrak{R}$ be the algebra of all relations on the set $\{\heartsuit, \diamondsuit\}$. Then take a, b, c to be given by their extensions

a	$\heartsuit$	$\diamondsuit$
$\heartsuit$	$+$	$+$
$\diamondsuit$	$-$	$-$

b	$\heartsuit$	$\diamondsuit$
$\heartsuit$	$-$	$-$
$\diamondsuit$	$+$	$-$

c	$\heartsuit$	$\diamondsuit$
$\heartsuit$	$-$	$+$
$\diamondsuit$	$+$	$+$

Then we get $a;b$, $-(c^{\smile})$, $c;b$, $-a$ and $-(a^{\smile})$ to be

$a;b$	$\heartsuit$	$\diamondsuit$
$\heartsuit$	$+$	$-$
$\diamondsuit$	$-$	$-$

$-(c^{\smile})$	$\heartsuit$	$\diamondsuit$
$\heartsuit$	$+$	$-$
$\diamondsuit$	$-$	$-$

$c;b$	$\heartsuit$	$\diamondsuit$
$\heartsuit$	$+$	$-$
$\diamondsuit$	$+$	$-$

$-a$	$\heartsuit$	$\diamondsuit$
$\heartsuit$	$-$	$-$
$\diamondsuit$	$+$	$+$

$-(a^{\smile})$	$\heartsuit$	$\diamondsuit$
$\heartsuit$	$-$	$+$
$\diamondsuit$	$-$	$+$

and it is clear that while $a; b \leq -(c^{\smile})$, we don't have $c; b \leq -(a^{\smile})$. Similarly, $a; b \leq -c$ (as $-c = -(c^{\smile})$) but $c; b \not\leq -a$. There doesn't seem to be any other natural candidate for a contraposing negation on this system.

Propositional Structures by `MaGIC` It is also possible to get propositional structures by sheer force of `MaGIC`. John Slaney has written a computer program which is a *Ma*trix *G*enerator for *I*mplication *C*onnectives [154]. For many logics (including any of our favourite logics) `MaGIC` will generate propositional structures up to 16 elements in size. It also searches for propositional structures that invalidate formulae supplied by a user. This can be very handy for quickly refuting non-theorems. Some `MaGICal` results are used elsewhere in this thesis.[1]

3.4 Conservative Extensions

In this section we will show another conservative extension result, due to Meyer and Slaney [80, 88]. We will prove that the positive parts of our logics are conservatively extended by negation. To do this, we need a number of preparatory results.

THEOREM 3.4 *Every* $\mathbf{DW}_C$*ps can be embedded in a complete* $\mathbf{DW}_C$*ps of its ideals — the power structure. If it is a* $\mathbf{TW}_C$*ps,* $\mathbf{EW}_C$*ps,* $\mathbf{L}_C$*ps,* $\mathbf{LI}_C$*ps, a* $\mathbf{C}_C$*ps or a* $\mathbf{CK}_C$*ps to start with, then so is the power structure.*

Proof: The complete structure is made up of the ideals of the original structure $\mathfrak{P}$. So, its elements are the sets $X \subset P$ where $x \in X$ and $y \leq x$ only if $y \in X$. If disjunction is present, we also require that $x, y \in X$ only if $x \vee y \in X$.

Conjunction on the complete structure is intersection, and if disjunction is present, $X \cup Y$ is the set $\{z : \exists x \exists y (x \in X, y \in Y \text{ and } z \leq x \cup y)\}$. We define fusion similarly, by requiring that $X \cdot Y = \{z : \exists x \exists y (x \in X, y \in Y \text{ and } z \leq xy\}$. If present, the residual is $X \Rightarrow Y = \{z : \exists x \exists y (x \in X, y \in Y \text{ and } z \leq x \Rightarrow y)\}$ (and similarly for the converse). Negation, if present $\overline{X} = \{z : \exists x (x \in X \text{ and } z \leq \overline{x})\}$. The e element of the complete structure is the principal ideal $\{x : x \leq e\}$. It is trivial to show that this structure satisfies the conditions required for a $\mathbf{DW}_C$ps. It is also complete in that it is closed under infinite joints (which are just intersections) and meets (which are just down-closure of arbitrary finite meets).

The extended properties of the original structure then go over to the complete structure. For example, if $b \cdot (a \cdot c) \leq a \cdot (b \cdot c)$ for each a, b and c, then $Y \cdot (X \cdot Z) = \{w : \exists x \exists y (x \in X, y \in Y, z \in Z \text{ and } w \leq y \cdot (x \cdot z)\} = \{z : \exists x \exists y (x \in X, y \in Y, z \in Z \text{ and } w \leq x \cdot (y \cdot z)\} = X \cdot (Y \cdot Z)$. The original structure is embedded in the larger one in the guise of its principal ideals. That is, the map $f : x \mapsto \{y : y \leq x\}$ is an injection of the original structure into its powerset. ◁

This construction helps us show that logics without $\rightarrow$, $\leftarrow$ or $\vee$ can be conservatively extended to the logics that contain them.

LEMMA 3.5 *In any power structure without disjunction, the union of two ideals is an ideal, and union satisfies the conditions for disjunction in a propositional structure.*

Proof: The union of any two downwardly closed sets is downwardly closed. As conjunction is intersection and disjunction is union, the distributive lattice properties are automatically satisfied. It is only needed to verify that $X \cdot (Y \cup Z) = X \cdot Y \cup X \cdot Z$. But this is simple, because if $w \in X \cdot (Y \cup Z)$ then $w \leq x \cdot v$ where $x \in X$ and $v \in Y \cup Z$. So either $v \in Y$ or $v \in Z$. Suppose without loss of generality that $v \in Y$. Then $x \cdot v \in X \cdot Y$ and so $w \in X \cdot Y \subseteq (X \cdot Y) \cup (X \cdot Z)$ as desired. Conversely, if $w \in (X \cdot Y) \cup (X \cdot Z)$ then without loss of generality, $w \in X \cdot Y$, and easily, $w \in X \cdot (Y \cup Z)$. The other distribution condition is just as simple to prove. ◁

So, if disjunction isn't present in the original structure, union will be its surrogate in the power structure. If it was present in the original structure, then it is extended to the infinitary case in the power structure by setting for any condition $C(-)$,

$$\bigcup_{C(Y)} Y = \{y : \text{for some } y_i \in Y_i \text{ where } C(Y_i) \text{ for } i = 1, \ldots, n;\ y \leq y_1 \cup \cdots \cup y_n\}$$

So, a disjunction of a set is the union of finite disjunctions of its members.

Lemma 3.6 *In any power structure,* $\left(\bigcup_{C(Y)} Y\right) \cdot X = \bigcup_{C(Y)}(Y \cdot X)$, *and similarly,* $X \cdot \bigcup_{C(Y)} Y = \bigcup_{C(Y)}(X \cdot Y)$.

Proof: Suppose that disjunction on the power structure is just union. Then if $z \in \left(\bigcup_{C(Y)} Y\right) \cdot X$, for some x and y, $z \leq y \cdot x$ where $y \in \bigcup_{C(Y)} Y$ and $x \in X$. Thus there is some Y such that $C(Y)$ where $y \in Y$. Then $y \cdot x \in Y \cdot X$ and hence $z \leq y \cdot x \in \bigcup_{C(Y)}(Y \cdot X)$ as desired. And if $z \in \bigcup_{C(Y)}(Y \cdot X)$ then $z \in Y \cdot X$ for some Y where $C(Y)$, giving $z \leq y \cdot x$ where $y \in Y$ and $x \in X$. Then clearly $z \in \bigcup_{C(Y)} Y \cdot X$.

If disjunction is not just union but the more complex construction given above, the reasoning is trickier. If $z \in \left(\bigcup_{C(Y)} Y\right) \cdot X$ we have for some $x \in X$ and $y \in \bigcup_{C(Y)} Y$, $z \leq y \cdot x$. This means that for some Y_1 to Y_n where $C(Y_i)$ for each i, $y \leq y_1 \cup \cdots \cup y_n$. Then $z \leq (y_1 \cup \cdots \cup y_n) \cdot x = (y_1 \cdot x) \cup \cdots \cup (y_n \cdot x) \in \bigcup_{C(Y)}(Y \cdot X)$ as desired. The converse argument is straightforward. This gives us the infinitary distribution laws we require. ◁

Lemma 3.7 *In any power structure without left or right residuals, but with disjunction, the functions*

$$Y \Leftarrow X = \bigcup_{Z \cdot X \leq Y} Z \qquad X \Rightarrow Y = \bigcup_{X \cdot Z \leq Y} Z$$

of X *and* Y *are the right and left residuals of* $\cdot$ *respectively.*

Proof: These are defined to satisfy the residuation conditions. For left residuation, note that if $Z \cdot X \leq Y$ then $Z \leq Y \Leftarrow X$ by definition. Conversely, note that $(Y \Leftarrow X) \cdot X = (\bigcup_{Z \cdot X \leq Y} Z) \cdot X = \bigcup_{Z \cdot X \leq Y} Z \cdot X \leq Y$. So, if $Z \leq Y \Leftarrow X$ then $Z \cdot X \leq Y$ as desired. ◁

Corollary *A logic without any of disjunction, standard or converse implication is conservatively extended by adding disjunction and each implication.*

Proof: Consider something not provable in the original vocabulary, in the original logic. There is some propositional structure that witnesses that fact. This structure can be embedded in a power structure in which disjunction and the residuals are defined. This is witness to the fact that the sequent is not provable in the extended logic either. $\triangleleft$

The situation with negation is a little trickier. We need to ensure that each proposition in our structure has an appropriate negation. There is no obvious way to cook up a negation in a power structure, so we need a different approach.

Definition 3.4 A propositional structure is said to be *rigorously compact* if there are elements T and F where for each a, $F \leq a \leq T$, $F \cdot a = F = a \cdot F$ and if $a \neq F$ then $T \cdot a = T = a \cdot T$.

THEOREM 3.8 *Every* $\mathbf{DW}^+$*ps can be embedded in a rigorously compact* $\mathbf{DW}^+$*ps. Furthermore, if it is a* $\mathbf{TW}^+$*ps,* $\mathbf{EW}^+$*ps,* $\mathbf{L}^+$*ps,* $\mathbf{LI}^+$*ps or* $\mathbf{C}^+$*ps, so is the rigorously compact structure in which it is embedded.*

Note that **CK** is not mentioned in this theorem. This is for the good reason that the result does not hold for **CK** propositional structures. As the top element T of a **CK**ps is the identity element e, the only way we can have $T \cdot a = T$ for $a \neq F$ is for a to equal T. So, the two element boolean algebra is the only rigourously compact **CK**ps. Now to the proof of the theorem.

Proof: Simple. Add extra top and bottom elements, and define fusion to satisfy the required conditions. Clearly no special conditions have been upset. Fusion remains associative or commutative or whatever under the new definition. For conditionals, take $F \Rightarrow a = a \Rightarrow T = a \Leftarrow F = T \Leftarrow a = T$ and $T \Rightarrow a = a \Leftarrow T = F$ unless $a = T$ and $a \rightarrow F = F \Leftarrow a = F$ unless $a = F$. This will satisfy residuation. This is the required structure. $\triangleleft$

LEMMA 3.9 *Any* $\mathbf{C}^+$*ps can be embedded into a* $\mathbf{C}$*ps. If the original structure is a* $\mathbf{CK}^+$*ps, so the large structure is a* $\mathbf{CK}$*ps.*

Proof: Take a $\mathbf{C}^+$ps on the set S (complete or rigourously compact, if necessary, to make sure that the structure has a greatest element T and a least element F.) and take another set T disjoint from S and of the same cardinality as S. Let $-$ be a bijection from S to T and back such that for any $x \in S \cup T$, $--x = x$. Now we define the other operations on $S \cup T$. Let $a, b \in S$ and $x, y \in S \cup T$.

$$a \cap' b = a \cap b \qquad a \cap' -b = -b \qquad -a \cap' b = -a \qquad -a \cap -b = -(a \cup b)$$

$$a \cup' b = a \cup b \qquad a \cup' -b = a \qquad -a \cup' b = b \qquad -a \cup -b = -(a \cap b)$$

$$a \cdot' b = a \cdot b \qquad a \cdot' -b = -(a \Rightarrow b) \qquad -a \cdot' b = -(b \Rightarrow a) \qquad -a \cdot' -b = F$$

$$x \Rightarrow' y = -(x \cdot' -y)$$

As Slaney and Martin report [80] it is tedious to check that the required postulates are satisfied. If the original structure was a $\mathbf{CK}^+$ps, then so is the new structure, as the top element is still the fusion identity. $\triangleleft$

This strategy won't work for **EW**, **TW** or **DW**, because we don't have the interderivability of fusion and implication. Instead, for **DW**, **TW** and **EW** we need a different strategy.

LEMMA 3.10 *Any* $\mathbf{DW}^+$*ps can be embedded into a* **DW***ps. If the original structure is a* $\mathbf{TW}^+$*ps or an* $\mathbf{EW}^+$*ps, the large structure is also a* **TW***ps or an* **EW***ps.*

Proof: Take a $\mathbf{DW}^+$ps and complete it and then rigourously compact it to result in a complete rigorously compact structure on some set S. Take another set T disjoint from S and of the same cardinality as S. Let $-$ be a bijection from S to T and back such that for any $x \in S \cup T$, $--x = x$. Now we define the other operations on $S \cup T$. Let $a, b \in S$.

$$a \cap' b = a \cap b \qquad a \cap' -b = -b \qquad -a \cap' b = -a \qquad -a \cap -b = -(a \cup b)$$

$$a \cup' b = a \cup b \qquad a \cup' -b = a \qquad -a \cup' b = b \qquad -a \cup -b = -(a \cap b)$$

$$a \cdot' b = a \cdot b \qquad a \cdot' -b = F \qquad -a \cdot' b = -\bigcup_{c \cdot b \leq a} c \qquad -a \cdot' -b = F$$

$$a \Rightarrow' b = a \Rightarrow b \qquad a \Rightarrow' -b = -b \qquad -a \Rightarrow' b = -I \qquad -a \Rightarrow' -b = b \Rightarrow a$$

It's trivial to check that this is a **DW**ps. The residuation conditions work by fiat, as do the structure preserving conditions for fusion.

To show that the new structure is a **TW**ps or an **EW**ps is a little more tricky. For the **EW** condition, note that $e \cdot -a = -a$ by design, so that condition is satisfied in the whole structure if it is in the original structure.

For the **TW** conditions we need to verify that $x \cdot (y \cdot z) \leq (x \cdot y) \cdot z$ for each x, y, z, and that contraposition holds. The latter is trivial (we have 'wired in' the contraposition rules). For the former, we consider the possibilities for each element being in S or T. If each are in S the condition is satisfied if satisfied in the original structure. Then many of these inequalities collapse into trivial identities. For example, $a \cdot (b \cdot -c) = F = (a \cdot b) \cdot -c$, and $a \cdot (-b \cdot -c) = F = (a \cdot -b) \cdot -c$, $-a \cdot (-b \cdot -c) = F = (-a \cdot -b) \cdot -c$, $-a \cdot (-b \cdot c) = F = (-a \cdot -b) \cdot c$. The only difficult case is with to show that $-a \cdot (b \cdot c) \leq (-a \cdot b) \cdot c$. Here, $-a \cdot (b \cdot c) = -\bigcup_{d \cdot (b \cdot c) \leq a} d$ and $(-a \cdot b) \cdot c = (-\bigcup_{d' \cdot b \leq a} d') \cdot c = -\bigcap_{d'' \cdot c \leq x} d''$, where $x = \bigcup_{d' \cdot b \leq a} d'$. So we wish to show that $-\bigcup_{d \cdot (b \cdot c) \leq a} d \leq -\bigcap_{d'' \cdot c \leq x} d''$, or equivalently that $\bigcap_{d'' \cdot c \leq x} d'' \leq \bigcup_{d \cdot (b \cdot c) \leq a} d$. And for this it is enough to show that if $d'' \cdot c \leq x$ then $d'' \leq \bigcup_{d \cdot (b \cdot c) \leq a} d$ and the properties of meet will do the rest. But This is simple, because if $d'' \cdot c \leq x = \bigcup_{d' \cdot b \leq a} d'$, it means that $(d'' \cdot c) \cdot b \leq a$, and hence, that $d'' \cdot (c \cdot b) \leq a$ too. This means that $d'' \leq \bigcup_{d \cdot (b \cdot c) \leq a} d$ as desired. This completes our proof, the other **TW** condition follows from our inequality, by contraposition and double negation. ◁

EXCURSUS: Others who work in the area define propositional structures without recourse to $\cdot$ or e [17, 94]. Our reliance on these operations is only due to their fundamental importance to the natural deduction system. It is possible to eliminate reference to $\cdot$ or e if that is desired. For $\cdot$, simply define logics in terms of their implicational properties. For e, work with a *truth filter* of true elements instead of a principal filter of elements

$\{x : e \le x\}$ as we have defined it. Our way of doing things results in no loss of logical generality because fusion and e can be reinstated in complete propositional structures. If fusion is absent, it can be defined in a complete propositional structure by setting $X \cdot Y = \bigcup_{Y \le X \Rightarrow Z} Z$. If e is absent, take in a complete propositional structure it re-emerges as the intersection of all elements in the truth filter.

For those that are philosophically minded, the elements of the propositional structures we consider can be taken to be propositions. For example, the two-valued boolean algebra can be thought of as containing two propositions, the True and the False. Of course, we may not think that there only two propositions, so we may admit more liberal propositional structures. Or, you can consider the elements of 'small' propositional structures to be equivalence classes of propositions, such that classes remain invariant under the connectives. So, while we may not think that the True and the False are the only propositions there are, we might think that conjunction and disjunction respect the equivalence classes on propositions that they define. Here we need not be particularly fussed about what propositions Actually Are. This is an interesting debate, but one which is independent of our present topic, which is to give an account of possible structures of the class of propositions, according to the different logics in view. $\Box$

3.5 Validities

This section is another late addition, sketching how we could model world validity. Corresponding to world bunches W are world propositions w. These are such that $w \le w^2$, $w \le e$, and perhaps $w \le a \cup \overline{a}$ for each a. Then it is simple to show that $\Sigma \vdash_W A$ if and only if for each interpretation I into a propositional structure, and for each world proposition w in that structure, whenever $w \le I(B)$ for each $B \in \Sigma$, then $w \le I(A)$ too. The soundness and completeness proof for this equivalence is quite trivial, using the methods we've already seen. Propositional structures so closely parallel the proof theoretic apparatus of the last chapter that they do not add anything enlightening to the account of world validity. For that, we must wait until the ternary relational semantics of Chapters 5 and 6.

3.6 Note

[1]`MaGIC` is freely available from John Slaney at the Automated Reasoning Project, CISR, Australian National University, Canberra 0200, Australia; email: `jks@arp.anu.edu.au`. It is also publicly `ftp`able from `arp.anu.edu.au` (`/ARP/MaGIC/magic.2.0.3.tar.Z`).

Chapter 4
Relational Structures

'Quantifiers' assemble in conferences, workshops and symposia
to thrash out common problems
and share research methods and findings.
— ROBERT P. SWIERENGA 'Clio and Computers' [156]

Propositional structures model valid inference by relating the syntax of an argument to a field of propositions. This is sufficient for arguments using propositional connectives, but it won't do to model the properties of quantification. The value of $\forall x F(x)$ is related to the values of each of the instances $F(t)$ for terms t, but this isn't reflected in an unadorned propositional structure.

4.1 Definitions

To remedy the deficiency, we must add a *domain of quantification* which contains the objects over which the quantifiers range, and an interpretation for each of the predicates and function symbols. Once we have this structure, we can interpret our formulae of the language.

However, we must be careful. Formulae can contain free variables. For example, $F(\nu_0, \nu_1) \wedge \exists \nu_0 (G(\nu_0))$ has ν_0 and ν_1 free. This formula has an interpretation only when we decide on the referents of its free variables. There are two policies for this: either free variables are treated as implicitly universally quantified, or they are treated as denoting individuals, which are to be chosen by other means. The logics we've studied take the latter approach. The former approach counts the inference from $F(x)$ to $\forall x F(x)$ as valid, but our logics take it to be invalid.

So, to interpret formulae with free variables we must decide what the free variables denote. We do this with a list $d = \langle d_0, d_1, d_2, \ldots \rangle$ so that d_n is the denotation of the variable ν_n when it occurs freely in a formula. This list is called a *valuation*. Then we can define the interpretation of $\forall \nu_n A$ *under the valuation* d to be the meet of the set of interpretations of A under the each of the valuations given by replacing the nth element of d by each element in the domain.

For this to work, the meets and joins of these sets must exist. It's well known that not all lattices contain the meets and joins of arbitrary sets, so more work has to be done to ensure that quantification can be modelled.

One way to do this is to insist that the lattice be complete. Anything falsifiable in a propositional structure can be falsified in a complete propositional structure of the appropriate kind. By demanding completeness of our structures we lose nothing at the propositional level, and we gain the ability to model quantifiers.

However, if we want to limit the size of a propositional structure, we may not want it to be complete. For example, the structure given by the rational numbers in the interval $[0, 1]$, using Łukasiewicz's interpretation of the propositional connectives, then forming a complete lattice that contains $[0, 1]$ as a sublattice means going from a structure of size $\aleph_0$ to one of size $2^{\aleph_0}$. This is more extravagant than most applications need. In a countable language, there are only a countable number of meets and joins that need to be evaluated, and so, only a countable number of extra propositions (at most) will need to be added to the propositional structure to ensure that the relevant meets and joins are defined. This is because in a countable language there is no way to express most of the offending meetless or joinless sets. The fact that they have no meet or join doesn't become relevant, because they do not arise in the behaviour of any formula in the language.

In this section we will demonstrate how we can get away without complete propositional structures, while still making sure that whenever we need a meet or a join of a set, it is there, ready for us to use. This account borrows heavily from ideas due to Ross Brady [17]. However, this work extends Brady's work to deal with languages with fusion or function symbols, and it recasts his ideas in different (and hopefully clear) notation.

We will define a field of functions that represent the formulae in our language. Then we only require generalised meets and joints to be defined for *these* functions. To characterise the class of functions we need a fair bit of technical machinery. We will work through the requirements one at a time.

Consider the interpretation of a formula A as a function taking a valuation as argument and giving an element of a propositional structure. This is a function from $D^\omega \to P$ where D is the domain of the relational structure and P is the domain of the propositional structure. We're interested in the class of all such functions: Call it $\mathcal{R}$.

Consider the formula A represented by the function R. If we exchange the free variables in A for other variables, the result is a different function R', which is also an element of $\mathcal{R}$. The exchange of free variables can be represented by a function $\sigma : \omega \to \omega$, such that we can exchange v_i for $v_{\sigma(i)}$ for each i. This function gives us a way to construct R' from R. The function σ lifts to a map $\rho : D^\omega \to D^\omega$ by setting $\rho(d_0, d_1, \ldots) = \langle d_{\sigma(0)}, d_{\sigma(1)}, \ldots\rangle$. Then it is easy to see that $R' = R\rho$, where $R\rho(d) = R(\rho(d))$.

Note that all exchanges like this must affect only a finite number of variables (as formulae only contain a finite number of terms, and terms only contain a finite number of free variables). So, all exchanges can be represented by functions $\rho : D^\omega \to D^\omega$ of the form described above, which depend on only a finite number of their arguments. (Such functions are called *constant almost everywhere*.) Call these functions ρ *exchange functions*. We have just shown that our set $\mathcal{R}$ must be closed under composition with exchange functions. If $R \in \mathcal{R}$ then $R\rho \in \mathcal{R}$ too, for any exchange function ρ.

Related to this is the possibility of replacing a free variable in A by an arbitrary term t. Consider a function $r : D^n \to D$ that models an n-place function symbol f from our language. Then replacing the variable v_m in A by the term $f(v_{m1}, v_{m2}, \ldots, v_{mn})$ results in a new formula A', and a new function representing that formula. Where the old function depended on d_m the new function will depend on $r(d_{m1}, d_{m2}, \ldots, d_{mn})$.

To determine the function representing the new formula, we need to keep a track of which variables occur within the new term. To do this, consider the function $r^+ : D^\omega \to D$ by setting $r^+(d) = r(d_0, d_1, \ldots, d_{n-1})$. (Note that the function r^+ is also constant almost everywhere.) Then if R represents A, and R' represents A', $R'(d)$ is found by replacing the mth value of d by $r(d_{m1}, d_{m2}, \ldots, d_{mn})$ and applying R to the result. This behaviour can be replicated by the following map. Let ρ be the exchange map that sends d_{mi} to d_{i-1} for $i = 1$ to n, and leaves every other d_k constant. Then define $[m, r^+, \rho] : D^\omega \to D^\omega$ as follows.

$$[m, r^+, \rho](d) = \langle d_0, d_1, \ldots, d_{m-1}, r^+(\rho(d)), d_{m+1}, \ldots \rangle$$

The function $[m, r^+, \rho]$ takes a list d and replaces the mth element of the list by $r^+(\rho(d))$. To make this kind of construction easier to write, we take $d(m/a)$ to be the result of replacing d_m in d by a. Then $[m, r^+, \rho](d) = d\big(m/r^+(\rho(d))\big)$. Then $R'(d) = R\big([m, r^+, \rho](d)\big)$, or more simply, $R' = R[m, r^+, \rho]$.

We call $[m, r^+, \rho]$ a *function injection*. We have shown that $\mathcal{R}$ is closed under composition with function injections, provided that these are based on functions r^+ that arise from representing functions in our language.

If R and S represent A and B respectively, then $A \to B$, $A \wedge B$, $A \vee B$, $\sim A$, $A \circ B$ and $A \leftarrow B$ must be represented by $R \Rightarrow S$, $R \cap S$, $R \cup S$, $\overline{R}$, $R \cdot S$ and $R \Leftarrow S$ each defined componentwise in the obvious way. So, $\mathcal{R}$ is closed under componentwise composition with respect to the functions on the propositional structure.

If R represents A, then $\bigcap_{a \in D} R\big(d(n/a)\big)$ represents $\forall v_n A$ and $\bigcup_{a \in D} R\big(d(n/a)\big)$ represents $\exists v_n A$. For this to make sense we need to say what $\bigcap$ and $\bigcup$ mean in this context. There are a number of conditions: they are *partial* functions from $\mathcal{P}P$ to P, such that for any $R \in \mathcal{R}$, $\bigcap_{a \in D} R\big(d(n/a)\big) \leq R(d) \leq \bigcup_{a \in D} R\big(d(n/a)\big)$. This means that $\bigcup$ and $\bigcap$ are *partial* lower and upper bounds respectively. To make them greatest lower and least upper bounds on our domains is to say that for any R and S in $\mathcal{R}$, if S is n-constant and $R(d) \leq S(d)$, $\bigcup_{a \in D} R\big(d(n/a)\big) \leq S(d)$ too. (S is n-constant when $S\big(d(n/a)\big) = S(d)$ for each $a \in D$ — if $S(d)$ doesn't depend on the nth element of d.) Similarly, if R is n-constant and $R(d) \leq S(d)$ then $R(d) \leq \bigcap_{a \in D} S\big(d(n/a)\big)$. For simplicity, we take $\bigcap_n R$ to be the function given by setting $\big(\bigcap_n R\big)(d) = \bigcap_{a \in D} R\big(d(n/a)\big)$. Similarly for $\bigcup_n R$. Then we have just shown that if $R \in \mathcal{R}$, so are $\bigcap_n R$ and $\bigcup_n R$ for each n.

Then we need $\bigcap$ and $\bigcup$ to interact with $\cup$ and $\cdot$ appropriately. If R is n-constant, then a number of distribution facts hold. Infinitary meet distributes over finitary join: $\bigcap_{a \in D}\big(R(d(n/a)) \cup S(d(n/a))\big) = R(d) \cup \bigcap_{a \in D} S\big(d(n/a)\big)$, and infinitary join distributes over fusion, on both sides: $\bigcup_{a \in D}\big(R(d(n/a)) \cdot S(d(n/a))\big) = R(d) \cdot \bigcup_{a \in D} S\big(d(n/a)\big)$, and $\bigcup_{a \in D}\big(S(d(n/a)) \cdot R(d(n/a))\big) = \big(\bigcup_{a \in D} S(d(n/a))\big) \cdot R(d)$.

This gives us a number of closure conditions for $\mathcal{R}$. But this isn't enough to ensure that $\mathcal{R}$ is populated: for this, we need a base condition.

If the n-ary predicate F is interpreted by the function $R : D^n \to P$ in the relational structure, then the formula $F(v_0, v_1, \ldots, v_{n-1})$ gives us the function $R^+ : D^\omega \to P$ by setting $R^+(d) = R(d_0, d_1, \ldots, d_{n-1})$. These functions R^+ are the ground elements in $\mathcal{R}$ for our recursive construction. (Note that the functions R^+ are, like the functions r^+, constant almost everywhere.) We have now done enough to motivate our definition.

Definition 4.1 Where $\mathfrak{P}$ is a propositional structure with domain P; given a set $\mathcal{R}_1$ of functions $D^\omega \to P$, constant almost everywhere, and a set $\mathcal{R}_2$ of functions $D^\omega \to D$, constant almost everywhere, then $\mathcal{R}$ is a *field of functions based on* $\mathcal{R}_1$ *and closed under* $\mathcal{R}_2$ with generalised meet and join operators $\bigcup, \bigcap$ if and only if the following conditions hold:

- $\mathcal{R}_1 \subseteq \mathcal{R}$.
- If $R, S \in \mathcal{R}$ then each relevant function from among $R \cap S$, $R \cup S$, $R \Rightarrow S$, $R \Leftarrow S$, $R \cdot S$ and $\overline{R}$, $\bigcap_n R$ and $\bigcup_n R$ is in $\mathcal{R}$.
- If ρ is an exchange function, then if $R \in \mathcal{R}$, so is $R\rho$.
- If $r \in \mathcal{R}_2$ and ρ is an exchange function, and $m \in \omega$, then $R[m, r, \rho] \in \mathcal{R}$ whenever $R \in \mathcal{R}$.
- $\bigcap, \bigcup$ are partial functions from $\mathcal{P}P$ to P
- If $R \in \mathcal{R}$, $\bigcap_{a \in D} R\big(d(n/a)\big) \leq R(d) \leq \bigcup_{a \in D} R\big(d(n/a)\big)$.
- If $R \in \mathcal{R}$, $S \in \mathcal{R}$ is n-constant and $R(d) \leq S(d)$, then $\bigcup_{a \in D} R\big(d(n/a)\big) \leq S(d)$.
- If $R \in \mathcal{R}$ is n-constant, $S \in \mathcal{R}$ and $R(d) \leq S(d)$ then $R(d) \leq \bigcap_{a \in D} S\big(d(n/a)\big)$.
- If R is n-constant, then $\bigcap_{a \in D}\big(R(d(n/a)) \cup S(d(n/a))\big) = R(d) \cup \bigcap_{a \in D} S(d(n/a))$, $\bigcup_{a \in D}\big(R(d(n/a)) \cdot S(d(n/a))\big) = R(d) \cdot \bigcup_{a \in D} S(d(n/a))$, and $\bigcup_{a \in D}\big(S(d(n/a)) \cdot R(d(n/a))\big) = \big(\bigcup_{a \in D} S(d(n/a))\big) \cdot R(d)$.

Given this, we can define our models for the predicate calculus.

Definition 4.2 A *relational structure* on a propositional structure $\mathfrak{P}$ is a structure

$$\mathfrak{A} = \big\langle D_{\mathfrak{A}}; \bigcup, \bigcap, R_\xi\ (\xi < \alpha_1),\ r_\xi\ (\xi < \alpha_2)\big\rangle$$

such that

- $D_{\mathfrak{A}}$ is non-empty.
- For each $\xi < \alpha_1$, $R_\xi : D_{\mathfrak{A}}^{\mu_2(\xi)} \to P$.
- For each $\xi < \alpha_2$, $r_\xi : D_{\mathfrak{A}}^{\mu_2(\xi)} \to D_{\mathfrak{A}}$.
- $\bigcap$ and $\bigcup$ are generalised meet and join operators on a field of functions based on $\{R_\xi^+ : \xi < \alpha_1\}$ and closed under $\{r_\xi^+ : \xi < \alpha_2\}$.

These relational structures interpret the predicates in the propositional structure. Each R_ξ is a function that determines the proposition arising from the corresponding predicate when applied to each part of the domain, and each r_ξ is a function that determines the denotation of the corresponding function symbol when applied to each part of the domain.

Definition 4.3 Each $\mathfrak{P}$-relational structure determines a Tarskian *interpretation* of the language $\mathcal{L}$ in the following way: The interpretation is given by recursively defining two functions $\mathfrak{A}(-)_-$ and $\mathfrak{A}^t(-)_-$ determined by the relational structure $\mathfrak{A}$, giving an element of $\mathfrak{P}$ for each formula and valuation, and a domain element from $\mathfrak{A}$ for each term and valuation, respectively. The clauses are as follows:

$$\mathfrak{A}^t(v_n)_d = d_n, \quad \mathfrak{A}^t\big(f_\xi(t_1, \ldots, t_{\mu_2(\xi)})\big)_d = r_\xi\big(\mathfrak{A}^t(t_1)_d, \ldots, \mathfrak{A}^t(t_{\mu_2(\xi)})_d\big),$$

$$\mathfrak{A}\big(F_\xi(t_1,\ldots,t_{\mu_1(\xi)})\big)_d = R_\xi\big(\mathfrak{A}^t(t_1)_d,\ldots,\mathfrak{A}^t(t_{\mu_1(\xi)})_d\big).$$

For each of the connectives in the language the relevant condition holds.

$$\mathfrak{A}(A\wedge B)_d = \mathfrak{A}(A)_d \cap \mathfrak{A}(B)_d, \quad \mathfrak{A}(A\vee B)_d = \mathfrak{A}(A)_d \cup \mathfrak{A}(B)_d,$$

$$\mathfrak{A}(A\to B)_d = \mathfrak{A}(A)_d \Rightarrow \mathfrak{A}(B)_d, \quad \mathfrak{A}(A\leftarrow B)_d = \mathfrak{A}(A)_d \Leftarrow \mathfrak{A}(B)_d,$$

$$\mathfrak{A}(\sim A)_d = \overline{\mathfrak{A}(A)_d}, \quad \mathfrak{A}(\bot)_d = F, \quad \mathfrak{A}(t)_d = e,$$

$$\mathfrak{A}(\forall v_n A)_d = \bigcap_{a\in D_\mathfrak{A}} (\mathfrak{A}(A))_{d(n/a)}, \quad \mathfrak{A}(\exists v_n A)_d = \bigcup_{a\in D_\mathfrak{A}} (\mathfrak{A}(A))_{d(n/a)}.$$

- Given the definition of an interpretation, we are free to define the notion of *satisfaction*: represented by $\vDash_d$, relativised to a valuation d. We take it that $\mathfrak{A}\vDash_d A$ if and only if $e\le\mathfrak{A}(A)_d$
- Clearly, if A is a sentence, the interpretation is independent of the valuation, and so is whether or not the sentence is satisfied in the relational structure. More generally, if the free variables of A are in $v_0,\ldots,v_n$, the interpretation of A depends only on the first $n+1$ elements of the valuation. In this case we will write $\mathfrak{A}(A)[d_0,\ldots,d_n]$ for $\mathfrak{A}(A)_d$ and $\mathfrak{A}\vDash A[d_0,\ldots,d_n]$ for $\mathfrak{A}\vDash_d A$.

For this definition of an evaluation to make sense, we need to prove that for each formula A, the relevant meets and joins $\bigcup\mathfrak{A}(A)$ and $\bigcap\mathfrak{A}(A)$ exist. To do this, we need only show that each function $\mathfrak{A}(A)_-$ appears in our field $\mathcal{R}$, because this field is closed under the required meets and joins. So, we prove the following lemma.

LEMMA 4.1 *Given a relational structure* $\mathfrak{A}=\langle D_\mathfrak{A};\mathcal{R},R_\xi\,(\xi<\alpha_1),\,r_\xi\,(\xi<\alpha_2)\rangle$, *for every formula* A, $\mathfrak{A}(A)_-\in\mathcal{R}$.

The proof of this lemma is almost contained within the conditions that $\mathcal{R}$ is required to satisfy. We have constructed $\mathcal{R}$ to satisfy just this result. The only difficulty in the proof is the base clause. But this is despatched by the following result.

LEMMA 4.2 *Take a relational structure* $\mathfrak{A}=\langle D_\mathfrak{A};\mathcal{R},R_\xi\,(\xi<\alpha_1),\,r_\xi\,(\xi<\alpha_2)\rangle$. *If* $\langle t_0,\ldots,t_n\rangle$ *is a list of terms in the language, then there is a function* s *formed by composing exchange functions and function injections* $[m,r^+,\rho]$ *such that for each* $d\in D^\omega$,

$$\mathfrak{A}^t(t_0)_d,\ldots,\mathfrak{A}^t(t_n)_d$$

are the first $n+1$ *terms of* $s(d)$.

Proof: By induction on the complexity of the list $\langle t_0,\ldots,t_n\rangle$, where the complexity is the number of occurrences of function symbols in the list. If there are no function symbols, we have a list of variables, and $\langle d_{i_0},\ldots d_{i_n}\rangle$ are the first $n+1$ terms of $\rho(d)$ for some appropriate exchange map ρ.

Now assume the hypothesis for all lists less complex than $\langle t_0,\ldots,t_n\rangle$. Consider a complex term t_i in that list. Either its outermost function symbol (say f_ξ) is of arity zero, or it is of greater arity. If it is of arity zero, consider the less complex list

$\langle t_0, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n \rangle$. By hypothesis, this is represented by some function s, made of exchanges and function injections. This means that

$$s(d) = \langle \mathfrak{A}^t(t_0)_d, \ldots, \mathfrak{A}^t(t_{i-1})_d, \mathfrak{A}^t(t_{i+1})_d, \ldots, \mathfrak{A}^t(t_n)_d, \text{irrelevant junk} \rangle$$

Then there is an exchange map ρ such that

$$(\rho s)(d) = \langle \mathfrak{A}^t(t_0)_d, \ldots, \mathfrak{A}^t(t_{i-1})_d, d_k, \mathfrak{A}^t(t_{i+1})_d, \ldots, \mathfrak{A}^t(t_n)_d, \text{irrelevant junk} \rangle$$

for some d_k. Which d_k is irrelevant for our purposes. Then $[i, r_\xi^+, id]$ will replace the ith term of this list with $r_\xi^+(d)$, the denotation of $f_\xi = t_i$. So

$$([i, r_\xi^+, id]\rho s)(d) = \langle \mathfrak{A}^t(t_0)_d, \ldots, \mathfrak{A}^t(t_n)_d, \text{irrelevant junk} \rangle$$

as desired. So, $[i, r_\xi^+, id]\rho s$ (which is composed of function injections and exchanges) is the function we are looking for (which is also composed of function injections and exchanges).

The other possibility is that f_ξ is of arity one or greater. Suppose t_i is the term $f_\xi(t_{i_0}, \ldots, t_{i_m})$ of some arity $m \geq 0$. Then by hypothesis, the less complex list

$$\langle t_0, \ldots, t_{i-1}, t_{i_0}, \ldots, t_{i_m}, t_{i+1}, \ldots, t_n \rangle$$

is represented by some function s composed of exchanges and function injections. Then consider ρ_1 which sends j to i_j for each $j = 0, \ldots, m$. Then $[i, r_\xi^+, \rho_1]$ replaces the ith element of d with $r_\xi(d_{i_0}, \ldots d_{i_m})$. This means that $[i, r_\xi^+, \rho_1]s$ represents the list $\langle t_0, \ldots, t_{i-1}, f_\xi(t_{i_0}, \ldots t_{i_m}), t_{i_1}, \ldots, t_{i_m}, t_{i+1}, \ldots, t_n \rangle$. Finally we apply an exchange that shifts out the irrelevant elements t_{i_1} to t_{i_m}. Take ρ_2 to be an exchange function that keeps elements 0 to i constant, and shifts the next m elements in the list to beyond the elements in the place of t_{i+1} to t_n, and shifts these elements to the places $i+1$ to n in the list. Then the function $\rho_2[i, r_\xi^+, \rho_1]s$ represents the original list.

This completes the proof by induction. ◁

Now the proof of the Lemma 4.1 is simple.

Proof: By induction on the complexity of A. If A is atomic, then it is of the form $F_\xi(t_1, \ldots, t_{\mu_1(\xi)})$. Now $\mathfrak{A}\big(F_\xi(t_1, \ldots, t_{\mu_1(\xi)})\big)_d = R_\xi\big(\mathfrak{A}^t(t_1)_d, \ldots, \mathfrak{A}^t(t_{\mu_1(\xi)})_d\big)$. By the previous lemma there is a function s composed of permutations and injections such that the first $\mu_1(\xi)$ terms of $s(d)$ are $\mathfrak{A}^t(t_1)_d$, up to $\mathfrak{A}^t(t_{\mu_1(\xi)})_d$ respectively. This means that $R_\xi^+\big(s(d)\big) = R_\xi\big(\mathfrak{A}^t(t_1)_d, \ldots, \mathfrak{A}^t(t_{\mu_1(\xi)})_d\big)$, which then shows that $\mathfrak{A}(A)_- \in \mathcal{R}$ as desired.

Once we can model atomic sentences, then connectives and quantifiers are dealt with by the closure conditions on $\mathcal{R}$. So, all functions $\mathfrak{A}(A)_-$ appear in our set $\mathcal{R}$. ◁

Now that we've gone to all of this trouble in defining structures, we had better put them to good use. We will show that relational structures actually extend propositional structures to give a sound and complete semantics for each of our logics.

THEOREM 4.3 [SOUNDNESS] *If $\Sigma \vdash A$ in* **DW** *then for each relational structure $\mathfrak{A}$ and valuation d on $\mathfrak{A}$, if $\mathfrak{A} \vDash_d B$ for each $B \in \Sigma$, then $\mathfrak{A} \vDash_d A$ also. Similarly, if $\Sigma \vdash A$ in any stronger logic then each relational structure $\mathfrak{A}$ based on an appropriate propositional structure, and for each valuation d on $\mathfrak{A}$, if for each $B \in \Sigma$, $\mathfrak{A} \vDash_d B$, then $\mathfrak{A} \vDash_d A$ also.*

Proof: For soundness it is sufficient to prove that the Hilbert-style axioms of **DW** come out as true under in each relational structure under each valuation, and that *modus ponens* is truth preserving. The result for the non-quantificational axioms and *modus ponens* follows from the result in the previous chapter, because the extra relational machinery doesn't enter in any essential way. For quantificational axioms, we will work through three, which give the flavour of the result.

For $\forall$ Elimination, suppose that t is free for ν_n in $A(\nu_n)$. Then $\mathfrak{A}(A(t))_d = \mathfrak{A}(A(\nu_n))_{d(n/\mathfrak{A}^t(t)_d)}$, so $\mathfrak{A}(\forall\nu_n A(\nu_n))_d = \bigcap_{a\in D_{\mathfrak{A}}} \mathfrak{A}(A(\nu_n)_{d(n/a)} \leq \mathfrak{A}(A(t))_d$. This means that $\mathfrak{A} \vDash_d \forall\nu_n A(\nu_n) \to A(t)$, as desired.

For $\forall\vee$ Distribution, suppose that ν_n is not free in A. Then $\mathfrak{A}(\forall\nu_n(A \vee B))_d = \bigcap_{a\in D_{\mathfrak{A}}} \mathfrak{A}(A \vee B)_{d(n/a)} = \mathfrak{A}(A)_d \cup \bigcap_{a\in D_{\mathfrak{A}}} \mathfrak{A}(B)_{d(n/a)} = \mathfrak{A}(A \vee \forall\nu_n B)_d$. So, $\mathfrak{A} \vDash_d \forall\nu_n(A \vee B) \to A \vee \forall\nu_n B$.

For $\exists$ Distribution, suppose that ν_n is not free in B. Then $\mathfrak{A}(\forall\nu_n(A \to B))_d = \bigcap_{a\in D_{\mathfrak{A}}} \mathfrak{A}(A \to B)_{d(n/a)}$. Similarly, $\mathfrak{A}(\exists\nu_n A \to B) = \bigcup_{a\in D_{\mathfrak{A}}} \mathfrak{A}(A)_{d(n/a)} \Rightarrow \mathfrak{A}(B)_d$. Then $\mathfrak{A} \vDash_d \forall\nu_n(A \to B) \to (\exists\nu_n A \to B)$ only if $\bigcup_{a\in D_{\mathfrak{A}}} \mathfrak{A}(A)_{d(n/a)} \cdot \bigcap_{a\in D_{\mathfrak{A}}} \mathfrak{A}(A \to B)_{d(n/a)} \leq \mathfrak{A}(B)$. But $\mathfrak{A}(A)_d \cdot \mathfrak{A}(A \to B)_d \leq \mathfrak{A}(B)_d$ and $\bigcup_{a\in D_{\mathfrak{A}}} \mathfrak{A}(A \to B)_{d(n/a)} \leq \mathfrak{A}(A \to B)$ gives $\mathfrak{A}(A)_d \cdot \bigcup_{a\in D_{\mathfrak{A}}} \mathfrak{A}(A \to B)_{d(n/a)} \leq \mathfrak{A}(B)_d$, and as $\mathfrak{A}(B)_d$ is n-constant, $\bigcup_{a\in D_{\mathfrak{A}}} (\mathfrak{A}(A)_d \cdot \bigcup_{a\in D_{\mathfrak{A}}} \mathfrak{A}(A \to B)_{d(n/a)}) \leq \mathfrak{A}(B)_d$. But $\bigcup_{a\in D_{\mathfrak{A}}} \mathfrak{A}(A \to B)_{d(n/a)}$ is also n-constant, so we can distribute the meet over the fusion, and get $\bigcup_{a\in D_{\mathfrak{A}}} \mathfrak{A}(A)_d \cdot \bigcup_{a\in D_{\mathfrak{A}}} \mathfrak{A}(A \to B)_{d(n/a)} \leq \mathfrak{A}(B)_d$.

Finally, if A is true in each model structure under each evaluation, so is $\forall x A$, so the metarule is preserved. ◁

Completeness will be proved by constructing a countermodel, just as in the previous chapter. As before, we will construct a model out of the language itself.

Definition 4.4 Given a set Σ of sentences, in one of the logics under discussion, the Σ *formula model* is a structure

$$\mathfrak{R}_\Sigma = \left\langle D_{\mathfrak{R}_\Sigma}; \bigcap, \bigcup, R_\xi\,(\xi < \alpha_1), r_\xi\,(\xi < \alpha_2)\right\rangle$$

where each $R_\xi : D_{\mathfrak{R}_\Sigma}^{\mu_1(\xi)} \to P$, and $r_\xi : D_{\mathfrak{R}_\Sigma}^{\mu_2(\xi)} \to D_{\mathfrak{R}_\Sigma}$, and where P is the domain of the Σ formula algebra given by the equivalence classes of Σ-equivalent formulae. $D_{\mathfrak{R}_\Sigma}$ is the set of all terms in the language, adding denumerably many constants $c_0, c_1, \ldots$ not in Σ, and closing these under function application. The functions and predicates are given by setting

$$r_\xi(d_1, \ldots, d_{\mu_2(\xi)}) = f_\xi(d_1, \ldots, d_{\mu_2(\xi)}) \qquad R_\xi(d_1, \ldots, d_{\mu_1(\xi)}) = [F_\xi(d_1, \ldots, d_{\mu_1(\xi)})]$$

We drop the use of subscripted Σ when the context is clear.

The Σ *formula evaluation* is a map from formulae and lists of terms to elements of the propositional algebra given by setting

$$\mathfrak{R}_\Sigma(A)_d = [A|_d]$$

where $A|_d$ is given by (simultaneously for each i) replacing the free occurrences of the variable v_i with d_i. The maps $\bigcup$ and $\bigcap$ are defined by setting

$$\bigcup_{a\in D}[A|_{d(n/a)}] = [\exists v_n A|_d] \qquad \bigcap_{a\in D}[A|_{d(n/a)}] = [\forall v_n A|_d]$$

for any formula A. (This is independent of the choice of A in the equivalence class, because of the substitution theorem.)

LEMMA 4.4 *For any Σ, $\mathfrak{R}_\Sigma$ is a relational structure, with evaluation $\mathfrak{R}_\Sigma(-)_-$.*

Then completeness is an immediate corollary.

COROLLARY [COMPLETENESS] *If $\Sigma \not\vdash A$ for some A, then there is a model $\mathfrak{R}_\Sigma$ and evaluation d that satisfies each element of Σ but not A. If $X \not\Vdash A$ for some A, then there is a model $\mathfrak{R}_\Sigma$ and evaluation d such that $\mathfrak{R}_\Sigma(X)_d \not\leq \mathfrak{R}_\Sigma(A)$.*

Proof: (Of the lemma.) We need to show that $\mathfrak{R}_\Sigma(-)_-$ satisfies the inductive definitions for an evaluation, that the base cases for the induction are satisfied, and that $\bigcup$ and $\bigcap$ are generalised meet and join operators on the closed field of functions.

The Σ formula algebra satisfies the required conditions for formula structures, by considerations from the previous chapter. For universal and existential quantifiers, the clauses are given by fiat. Also, the base cases for the induction are trivial. By definition, $\mathfrak{R}^t(t)_d = t|_d$ and $\mathfrak{R}(F_\xi(t_1, \ldots, t_{\mu_1(\xi)}))_d = [F_\xi(t_1, \ldots, t_{\mu_1(\xi)})|_d]$.

So, we need just show that $\bigcap$ and $\bigcup$ satisfy the required conditions. But this is simple. Firstly,

$$\bigcap_{a\in D}[A|_{d(n/a)}] \leq [A|_d] \leq \bigcup_{a\in D}[A|_{d(n/a)}]$$

follow immediately from the theorems $\forall v_n A(v_n) \to A(t)$ and $A(t) \to \exists v_n A$ (where t free for v_n in A). For the other conditions, suppose that $[A|_-]$ is n-constant. and that $[A|_d] \leq [B|_d]$. We wish to show that $[A|_d] \leq \bigcup_{a\in D_\Sigma}[B|_{d(n/a)}] = [\exists v_n B|_d]$. But this is simple. Indicate the free instances of v_n in B by writing it $B(v_n)$. Then we have $\Sigma \vdash A \to B(t)$ for any t free for v_n in $B(v_n)$. So, in particular $\Sigma \vdash A \to B(c_m)$ for some c_n not appearing in Σ or A. Take the Hilbert style proof of $A \to B(c_m)$ from Σ. Replace all instances of c_m with a variable v_k not occurring freely in the premises used in the proof (there must be some, as only a finite number of premises are used in a proof). We have a proof of $\Delta \vdash A \to B(v_k)$. As a result, by applying conditional proof, we have an arbitrary fusion F of Δ such that $\vdash F \to (A \to B(v_k))$. But now this means that $\vdash \forall v_k(F \to (A \to B(v_k))$ and by $\forall \to$ distribution, $\vdash F \to (A \to \forall v_k B(v_k))$ giving $\Sigma \vdash A \to \forall v_k B(v_k)$ as desired. The other cases are dual.

For logics extending **DW**, the extra axioms are preserved in the appropriate model structures, so the soundness proof is no more difficult. ◁

Now we have structures that model deduction in our logics. We are free to use propositional structures to model quantification in the obvious way. Our next task is to prove some of the usual classical results that compare structures.

4.2 Comparing Structures

The Löwenheim-Skolem theorems concern relations between relational structures. In order to prove extensions to the classical theorems to the context of our logics, we need to extend the classical definitions of the relationships between relational structures. The definition below gives *some* of the relations that may hold between relational structures. In giving these definitions and proving theorems with them, we will follow the general presentation of Bell and Slomson [14].

Definition 4.5

- Given two relational structures $\mathfrak{A} = \langle D_{\mathfrak{A}}; \bigcup, \bigcap, R_\xi\,(\xi < \alpha_1), r_\xi\,(\xi < \alpha_2)\rangle$ and $\mathfrak{B} = \langle D_{\mathfrak{B}}; \bigcup', \bigcap', S_\xi\,(\xi < \alpha_1), s_\xi\,(\xi < \alpha_2)\rangle$ of the same type and on the same propositional structure, $\mathfrak{A}$ is a *substructure* of $\mathfrak{B}$ and $\mathfrak{B}$ is an *extension* of $\mathfrak{A}$ if $D_{\mathfrak{A}} \subseteq D_{\mathfrak{B}}$, if $\bigcup = \bigcup'$ and $\bigcap = \bigcap'$, and if each of the functions R_ξ and r_ξ are the restrictions of the corresponding functions S_ξ and s_ξ to A. (Note that in the case of the functions r_ξ and s_ξ we need to make sure that each r_ξ is actually a function into the domain A.) We write this relation as '$\mathfrak{A} \subseteq \mathfrak{B}$'.
- In particular, to each nonempty subset X of $D_{\mathfrak{B}}$ there corresponds the substructure $\langle X; S_\xi \cap (X^{\mu_1(\xi)} \times P)\,(\xi < \alpha_1), s_\xi \cap X^{\mu_1(\xi)+1}\,(\xi < \alpha_2)\rangle$ of $\mathfrak{B}$, given that each $s_\xi \cap X^{\mu_1(\xi)+1}$ is a total function. (In this construction, intersecting S_ξ with $X^{\mu_1(\xi)} \times P$ means reducing the functions from with domain $D_{\mathfrak{B}}^{\mu_1(\xi)}$ and range P to functions with domain $X^{\mu_1(\xi)}$, and the same range.) We call this substructure the *restriction* of $\mathfrak{B}$ to X, and write this as '$\mathfrak{B} \restriction X$'.
- An *isomorphism* between structures $\mathfrak{A}$ and $\mathfrak{B}$ is given by a one-one and onto map $h : D_{\mathfrak{A}} \to D_{\mathfrak{B}}$ such that for each ξ, $R_\xi(d_1, \ldots, d_{\mu_1(\xi)}) = S_\xi(h(d_1), \ldots, h(d_{\mu_1(\xi)}))$ and $r_\xi(d_1, \ldots, d_{\mu_2(\xi)}) = s_\xi(h(d_1), \ldots, h(d_{\mu_2(\xi)}))$ If there is an isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$ we indicate this by writing '$\mathfrak{A} \simeq \mathfrak{B}$'.
- Clearly, if $\mathfrak{A} \simeq \mathfrak{B}$ then for every sentence A, $\mathfrak{A}(A) = \mathfrak{B}(A)$. Almost as clearly, the converse of this does not hold, as in the classical case. As in the classical case, we will say that if $\mathfrak{A}$ and $\mathfrak{B}$ give the same sentences the same values, then $\mathfrak{A}$ and $\mathfrak{B}$ are *elementarily equivalent*. This is written '$\mathfrak{A} \equiv \mathfrak{B}$'.
- As in the classical case, elementary equivalence is not so easy to work with, as it deals with sentences alone, and not formulae, making proofs by induction on the complexity of sentences quite difficult. If $\mathfrak{A} \subseteq \mathfrak{B}$, and if for each $d \in D_{\mathfrak{A}}^{\omega}$, $\mathfrak{A}(A)_d = \mathfrak{B}(A)_d$, we say that $\mathfrak{B}$ is an *elementary extension* of $\mathfrak{A}$ and that $\mathfrak{A}$ is an *elementary substructure* of $\mathfrak{B}$. This is written $\mathfrak{A} \prec \mathfrak{B}$. An embedding h of $\mathfrak{A}$ into $\mathfrak{B}$ is an *elementary embedding* of $\mathfrak{A}$ into $\mathfrak{B}$ if for each $A(v_0, \ldots, v_n)$ and for any $a_0, \ldots, a_n$ in A, $\mathfrak{A}(A)[a_0, \ldots, a_n] = \mathfrak{B}(A)[h(a_0), \ldots, h(a_n)]$.

The definition of elementary equivalence is not the immediately obvious one, which would take two structures to be elementarily equivalent just when they make the same formulae true (without requiring that they have the same values in the propositional structure). This would make the next lemma quite difficult to prove, as the induction step for negation will fail. Whether or not $\mathfrak{A} \vDash_d A$ is not enough information to prove whether or not $\mathfrak{A} \vDash_d {\sim}A$ in *any* of our logics. So a stricter definition of equivalence is needed here. That's OK, as the upward/downward theorems show that models exist, and if we have an elementary substructure of a model, in our strong sense, it will also be an elementary substructure in the weaker sense.

Lemma 4.5 *Suppose* $\mathfrak{A} \subseteq \mathfrak{B}$. *Then (i)* $\mathfrak{A} \prec \mathfrak{B}$ *iff whenever* A *is a formula of* $\mathcal{L}$ *and* $d \in D_{\mathfrak{A}}^{\omega}$, *condition* (α_1) *and* (α_2) *holds*

$$\bigcup_{b \in D_{\mathfrak{B}}} \mathfrak{B}(A)_{d(n/b)} = \bigcup_{a \in D_{\mathfrak{A}}} \mathfrak{B}(A)_{d(n/a)} \tag{α_1}$$

$$\bigcap_{b \in D_{\mathfrak{B}}} \mathfrak{B}(A)_{d(n/b)} = \bigcap_{a \in D_{\mathfrak{A}}} \mathfrak{B}(A)_{d(n/a)} \tag{α_2}$$

(ii) If negation is present, either of (α_1) *and* (α_2) *clauses follows from the other.*

Proof: For (i), assume that $\mathfrak{A} \prec \mathfrak{B}$ and A is a formula of $\mathcal{L}$ with $d \in A^{\omega}$. Then we have $\bigcup_{b \in D_{\mathfrak{B}}} \mathfrak{B}(A)_{d(n/b)} = \mathfrak{B}\big((\exists v_n)A\big)_d$ by the interpretation of the existential quantifier. But $\mathfrak{B}\big((\exists v_n)A\big)_d = \mathfrak{A}\big((\exists v_n)A\big)_d$ by the hypothesis that $\mathfrak{A} \prec \mathfrak{B}$, but $\mathfrak{A}\big((\exists v_n)A\big)_d = \bigcup_{a \in D_{\mathfrak{A}}} \mathfrak{A}(A)_{d(n/a)}$, which is equal to $\bigcup_{a \in D_{\mathfrak{A}}} \mathfrak{B}(A)_{d(n/a)}$ by the fact that $\mathfrak{A} \prec \mathfrak{B}$ and that $d(n/a) \in D_{\mathfrak{A}}^{\omega}$. An identical argument gives the second condition.

Conversely, suppose that $\mathfrak{A} \subseteq \mathfrak{B}$, and that conditions (α_1) and (α_2) hold. We prove that for each $d \in D_{\mathfrak{A}}^{\omega}$, for every formula A in $\mathcal{L}$

$$\mathfrak{A}(A)_d = \mathfrak{B}(A)_d \tag{β}$$

by induction on the complexity of A. Firstly (β) holds for atomic formulae, and all of the terms get the same denotations, as $\mathfrak{A} \subseteq \mathfrak{B}$. And for conjunctions, disjunctions and implications (or any construction that receives a value that is a function of the values of its subformulae) (β) follows from the hypothesis that it holds for its subformulae. It remains to deal with the quantifiers. For $A = (\exists v_n)B$ where the hypothesis holds for B, we have $\mathfrak{A}\big((\exists v_n)B\big)_d = \bigcup_{a \in D_{\mathfrak{A}}} \mathfrak{A}(B)_{d(n/a)} = \bigcup_{a \in D_{\mathfrak{A}}} \mathfrak{B}(B)_{d(n/a)}$ (by the induction hypothesis for B) and by (α_1), $\bigcup_{a \in D_{\mathfrak{A}}} \mathfrak{B}(B)_{d(n/a)} = \bigcup_{b \in D_{\mathfrak{B}}} \mathfrak{B}(B)_{d(n/b)}$, but this is just $\mathfrak{B}\big((\exists v_n)B\big)_d$ as desired. The case for the universal quantifier is identical.

For (ii), if negation is present, the interdefinability of $\exists$ and $\forall$ show that (α_1) gives (α_2) and *vice versa*. ◁

Corollary *If* $\mathfrak{A} \subseteq \mathfrak{B}$ *then* $\mathfrak{A} \prec \mathfrak{B}$ *if for each formula* $A(v_0, \ldots, v_n)$ *with free variables in* $v_0, \ldots, v_n$, *and for each* $a_0, \ldots, a_{n-1} \in D_{\mathfrak{A}}$ *and each* $b \in D_{\mathfrak{B}}$ *there is an* $a \in D_{\mathfrak{A}}$ *such that* $\mathfrak{B}(A)[a_0, \ldots, a_{n-1}, b] = \mathfrak{B}(A)[a_0, \ldots, a_{n-1}, a]$.

Proof: If the conditions hold, then clearly (α_1) and (α_2) must hold, so $\mathfrak{A} \prec \mathfrak{B}$. ◁

4.3 Going Down

If you noticed the way that the model structures were defined for the completeness proof, you will see that we have a pretty strong form of the downward Löwenheim-Skolem theorem.

THEOREM 4.6 THE DOWNWARD LÖWENHEIM-SKOLEM THEOREM 1 *Let Σ be a set of sentences in a language $\mathcal{L}$ with α terms (for some cardinal α) and if in one of our logics, for some A, $\Sigma \not\vdash A$, then Σ has a model of size at most α, in that logic.*

Proof: The proof of the completeness theorem constructs a model with a domain consisting of all terms, together with a denumerable set of constants. There are at least denumerably many terms (the variables are terms) so the domain of this model is also of size α. This model satisfies Σ and not A as desired. ◁

However, there is more to a downward theorem than this. We can show when relational structures have properly smaller elementary substructures. This will show that such small models are not just syntactic constructs, but that large models of theories — perhaps the 'intended' model — contain elementary substructures that model the theory just as well.

THEOREM 4.7 *Let $\mathfrak{A}$ be an infinite relational structure of cardinal α in a language of cardinal ρ on a propositional structure $\mathfrak{P}$ of cardinal δ. Let C be a subset of $D_{\mathfrak{A}}$ of cardinal γ. If β is a cardinal satisfying $\gamma, \delta, \rho \leq \beta \leq \alpha$ then $\mathfrak{A} \restriction C$ has an extension $\mathfrak{B}$ of cardinal β which is an elementary substructure of $\mathfrak{A}$.*

Proof: Well order A by some order $<$. Define a sequence $\langle B_n : n < \omega \rangle$ of subsets of A as follows. B_0 is an arbitrary subset of $D_{\mathfrak{A}}$ containing C, and of cardinal β. Given B_n, B_{n+1} is found by adding new elements as follows: Take any formula $A(v_0, \ldots, v_m)$ in $\mathcal{L}$ with free variables in $v_0, \ldots, v_m$, and take $a_0, \ldots, a_{m-1}$ in B_n. For each $b \in D_{\mathfrak{A}}$, we add to B_{n+1} the $<$-least $a \in D_{\mathfrak{A}}$ such that $\mathfrak{A}(A)[a_0, \ldots, a_{m-1}, a] = \mathfrak{A}(A)[a_0, \ldots, a_{m-1}, b]$. Then, close this set under the functions in the structure $\mathfrak{A}$. As there are only δ possible values of $\mathfrak{A}(A)[a_0, \ldots, a_{m-1}, a]$, there are only ρ possible formulae A to consider and only ρ possible functions to close under, each B_n is no bigger than $\max(\delta, \gamma, \rho, \beta) = \beta$. Take $D_{\mathfrak{B}} = \bigcup_{n \in \omega} B_n$, and let $\mathfrak{B} = \mathfrak{A} \restriction D_{\mathfrak{B}}$. As $C \subseteq D_{\mathfrak{B}}$, $\mathfrak{B}$ is an extension of $\mathfrak{A} \restriction C$. It remains to prove that $\mathfrak{B} \prec \mathfrak{A}$.

Suppose $A(v_0, \ldots, v_n)$ is a formula of $\mathcal{L}$, that $a_0, \ldots, a_{n-1} \in D_{\mathfrak{B}}$, and that $a \in D_{\mathfrak{A}}$. We want to show that for some $b \in D_{\mathfrak{B}}$, $\mathfrak{A}(A)[a_0, \ldots, a_{n-1}, a] = \mathfrak{A}(A)[a_0, \ldots, a_{n-1}, b]$ Then the corollary to Lemma 4.5 gives us the result. But for each $0 \leq i < n$, $a_i \in D_{\mathfrak{B}}$ so there is some m where each $a_i \in B_m$. In this case there is a $b \in B_{m+1}$ where $\mathfrak{A}(A)[a_0, \ldots, a_{n-1}, a] = \mathfrak{A}(A)[a_0, \ldots, a_{n-1}, b]$. As $b \in D_{\mathfrak{B}}$, we have our result. ◁

COROLLARY THE DOWNWARD LÖWENHEIM-SKOLEM THEOREM 2 *Let Σ be a set of sentences of cardinal α, with an infinite model $\mathfrak{A}$ of cardinal $\geq \alpha$ in a propositional structure of size δ. If α is infinite, Σ has a model of cardinal $\max(\alpha, \delta)$, and otherwise, Σ has a model of cardinal $\max(\aleph_0, \delta)$. The domain of this model is a subset of the domain of $\mathfrak{A}$.*

Note here that the bound is not given by the size of the language, but by the size of the language used in Σ.

Proof: Let $\beta = \max(\aleph_0, \alpha)$ and $\gamma = \max(\aleph_0, \alpha, \delta)$. At most γ predicate letters and function symbols occur in Σ and so, we can regard Σ as a set of sentences of a language $\mathcal{L}'$ containing just these predicate letters and function symbols. It follows that the cardinal of $\mathcal{L}'$ is β (there are $\aleph_0$ variables regardless of whatever else is in $\mathcal{L}'$). $\mathfrak{A}$ is a model of Σ of cardinal $\geq \beta$, which exists, by hypothesis. By disregarding the relations corresponding to the predicate letters not occurring in $\mathcal{L}'$ we have a structure $\mathfrak{A}'$ of $\mathcal{L}'$ of cardinal $\geq \gamma$ which is a model of Σ. By the previous theorem $\mathfrak{A}'$ has an elementary substructure $\mathfrak{B}'$ of size γ. As $\mathfrak{B}' \prec \mathfrak{A}'$, $\mathfrak{B}'$ is also a model of Σ. By adding arbitrary relations to $\mathfrak{B}'$ as values of the predicate letters of $\mathcal{L}$ not in Σ, we have a structure $\mathfrak{B}$ in language $\mathcal{L}$ of cardinal γ which is a model of Σ. ◁

This result is tight, but it could be tighter. For example, the restriction to the size of the propositional structure is enough to make this result useless for reducing the domain of models with $[0,1]$ as a propositional structure down to countability, for example. There is one way around this restriction, provided that the offending large structure is complete, and $\bigcup$, $\bigcap$ are the genuine total meet and join function. (This restriction to complete propositional structures is not too bad. In the cases when we have a large propositional structure in mind, it is more than likely $[0,1]$ or some other complete structure, so our result applies.) We can use a *dense* substructure of the propositional structure in this way:

Definition 4.6 A set X is *dense* in a propositional structure $\mathfrak{P}$ with domain P just when $X \subseteq P$, and for each $x, z \in P$ where $x < z$ there is a $y \in X$ where $x \leq y \leq z$.

For example, the rationals between 0 and 1 are dense in $[0,1]$. And the rationals are a *countable* dense subset of $[0,1]$. This fact can be very helpful.

LEMMA 4.8 *Given a complete lattice $\mathfrak{L} = \langle L, \leq \rangle$ and a dense set X in $\mathfrak{L}$, of infinite cardinality δ', then for each set $Y \subseteq P$ there is a $Z \subseteq Y$ of cardinality no greater than δ' such that $\bigcup Z = \bigcup Y$ and $\bigcap Z = \bigcap Y$.*

Proof: Well order the set Y. Construct the set Z in stages. For Z_0, let z_0 be the first element in Y under the well ordering, and set $Z_0 = \{z_0\}$. Given Z_α, if $\bigcup Z_\alpha \neq \bigcup Y$, then let $z^+_{\alpha+1}$ be the first element of Y such that $z^+_{\alpha+1} \not\leq \bigcup Z_\alpha$ (which must exist). If $\bigcap Z_\alpha \neq \bigcap Y$, let $z^-_{\alpha+1}$ be the first element of Y such that $z^-_{\alpha+1} \not\geq \bigcap Z_\alpha$ (which must exist). If either of these cases obtain, continue the process up the ordinal hierarchy. For limits, take the union of all previous stages. This process has a limit, as Y is a set. Take the limit to be Z. Clearly it satisfies the meet and join conditions.

We prove that Z is at most of size δ'. If the set $\{z_0, z^+_1, \ldots, z^+_\alpha, \ldots\}$ is no larger than δ' we are home. Take the elements u_α to be defined in terms of the z^+_αs as follows. Firstly $u_0 = z_0$, $u_{\alpha+1} = u_\alpha \cup z^+_{\alpha+1}$, and for limit ordinals β, $u_\beta = \bigcup_{\alpha<\beta} u_\alpha$. By the definition of the z^+_αs, each u_α is distinct. They also form a chain in the lattice. Therefore, between any adjacent pair in the chain, there is an element of the dense subset X. That is, the set $\{u_0, u_1, \ldots\}$ is no larger than δ'. So, the sets $\{z_0, z_1^+, \ldots, z_\alpha^+, \ldots\}$ and Z are also no larger than δ'. ◁

Then we can tighten the result of Theorem 4.7.

THEOREM 4.9 *Let $\mathfrak{A}$ be an infinite relational structure of cardinal α in a language of cardinal ρ on a complete propositional structure $\mathfrak{P}$ of cardinal δ, with a dense subset X of cardinal δ'. Let C be a subset of $D_{\mathfrak{A}}$ of cardinal γ. If β is a cardinal satisfying $\gamma, \delta', \rho \leq \beta \leq \alpha$ then $\mathfrak{A} \restriction C$ has an extension $\mathfrak{B}$ of cardinal β which is an elementary substructure of $\mathfrak{A}$.*

Proof: Exactly as before, except for the additions to each B_n. The procedure for these additions is as follows: for each formula A and elements $a_0, \ldots, a_{n-1} \in B_n$, consider the set of all elements of $\mathfrak{P}$ that feature as values $\mathfrak{A}(A)[a_0, \ldots, a_{n-1}, b]$ as b ranges over A. Let this set be Y. We know that it has a subset Z of size at most δ' such that $\bigcup Z = \bigcup Y$ and $\bigcap Z = \bigcap Y$. For each element z of Z we add to B_{n+1} the $<$-first element of the domain a that satisfies $\mathfrak{A}(A)[a_0, \ldots, a_{n-1}, a] = z$. The (α) condition holds for $B \subseteq A$ by the same argument, and by our definition of Z.

At each stage we only add at most β elements, so $\bigcup_{n<\omega} B_n$ is of size β, as desired. ◁

COROLLARY THE DOWNWARD LÖWENHEIM-SKOLEM THEOREM 3 *Let Σ be a set of sentences of $\mathcal{L}$ of cardinal α, with an infinite model of cardinal $\geq \alpha$ in a propositional structure of size δ with a dense subset of cardinal δ'. If α is infinite, Σ has a model of cardinal $\max(\alpha, \delta')$, and otherwise, Σ has a model of cardinal $\max(\aleph_0, \delta')$.*

Proof: Exactly as before, but making use of Theorem 4.9 instead of Theorem 4.7. ◁

There is another way to get by the restriction in terms of the cardinality of the propositional structure involved. This relies on the possibility that not all of the propositional structure is used in evaluating truth in the relational structure. This definition forms the kernel of the idea.

Definition 4.7 The *image* of a relational structure $\mathfrak{A} = \langle D_{\mathfrak{A}}; \bigcup, \bigcap, R_\xi\ (\xi < \alpha_1), r_\xi\ (\xi < \alpha_2) \rangle$ in its propositional structure $\mathfrak{P}$ is the set of all propositions that are used in evaluating formulae from the relational structure. So, formally the image is the set of propositions $\{x : \text{for some formula } A \text{ and some } d \in D_{\mathfrak{A}},\ x = \mathfrak{A}(A)_d\}$.

The image of a relational structure is significant. It is a substructure of the propositional structure — it is closed under each operation on the propositional structure.

LEMMA 4.10 *Images are closed under each operation in the propositional structure. Thus, images are propositional structures in their own right, given that they inherit the operations from their 'parent.'*

Proof: Let I be an image. If $x = \mathfrak{A}(A)_{d_1} \in I$ then $\overline{x} = \mathfrak{A}(\sim A)_{d_1} \in I$. If $y = \mathfrak{A}(B)_{d_2} \in I$, then consider the free variables in A and in B. Reletter A and B so that the free variables in A are v_0 to v_m and those in B are v_{m+1} to v_n for some m and n. Call the resulting formulae A' and B'. Then apply exchanges to the valuations d_1 and d_2 to mirror this relettering, giving d_1' and d_2' where $x = \mathfrak{A}(A')_{d_1'}$ and $y = \mathfrak{A}(B')_{d_2'}$. Now the value of

$x = \mathfrak{A}(A')_{d'_1}$ depends only on the first $m+1$ values of d'_1 and the value of $y = \mathfrak{A}(B')_{d'_2}$ depends only on the next $n-m$ values of d'_2. Let d be another valuation which takes the first $m+1$ values d'_1, concatenates them with the next $n-m$ values of d'_2 and then adds whatever other junk you like. It is clear that $x = \mathfrak{A}(A')_d$ and $y = \mathfrak{A}(B')_d$. Then $x \cap y = \mathfrak{A}(A' \wedge B')_d \in I$, $x \cup y = \mathfrak{A}(A' \vee B')_d \in I$, $x \circ y = \mathfrak{A}(A' \cdot B')_d \in I$, $x \Rightarrow y = \mathfrak{A}(A' \to B')_d \in I$ and so on, for any other connective. Finally, $e = \mathfrak{A}(t)_d$ and $F = \mathfrak{A}(\perp)_d$ (if present). $\triangleleft$

So, any image is a propositional structure. It is also clear that a relational structure can be construed as being based on its image instead of the original propositional structure. We have the following:

Lemma 4.11 *A relational structure* $\mathfrak{A} = \left\langle D_{\mathfrak{A}}; \bigcup, \bigcap, R_\xi\,(\xi < \alpha_1), r_\xi\,(\xi < \alpha_2)\right\rangle$ *on a propositional structure* $\mathfrak{P}$ *is also a structure based on its image.*

Now we can tighten the conditions on the downward theorem.

Corollary The Downward Löwenheim-Skolem Theorem 4 *Let* Σ *be a set of sentences of cardinal* α*, with an infinite model* $\mathfrak{A}$ *of cardinal* $\geq \alpha$ *with an image of size* δ*. If* α *is infinite,* Σ *has a model of cardinal* $\max(\alpha, \delta)$*, and otherwise,* Σ *has a model of cardinal* $\max(\aleph_0, \delta)$*. The domain of this model is a subset of the domain of* $\mathfrak{A}$*.*

Proof: Simple. Apply the Downward Theorem number 2 to the relational structure based on the image, to get the smaller model. This can be construed as based on either the image or its parent. $\triangleleft$

4.4 Identity

To prove the upward results, we have to give a definition of identity in our models. Without this, proving an upward theorem is too easy. Take a relational structure of size α, pick an element in the domain and clone it β times (for any $\beta > \alpha$) such that the clones enter into exactly the same relations as the original. Voila! An elementary extension of the original structure, of size β. This is not a particularly perspicuous theorem. Each of the cloned elements are *equal* as far as the language is concerned. They are indiscernible from an external perspective. To give a more interesting result, then, we need to have a look at identity in the language.

Definition 4.8 Single out a predicate F_0 in a language $\mathcal{L}$, such that $\mu_1(0) = 2$. We will say that F_0 is a *weak identity predicate* in the model $\mathfrak{A}$ if the following conditions obtain for each term t_0 and t_1, each formula A and each valuation d.

- $\mathfrak{A} \vDash_d F_0(t_0, t_0)$,
- If $\mathfrak{A} \vDash_d F_0(t_0, t_1)$ then $\mathfrak{A}\big(A(t_0)\big)_d = \mathfrak{A}\big(A(t_1)\big)_d$.

In other words, objects are self-identical, and the substitution theorem holds in the strong sense.

These conditions are not particularly neat to check. However, they seem to be the right ones in our context. We do not wish to force that either $a = b$ or $a \neq b$ turn out true because excluded middle fails in each of our favourite logics. There seems to be no principled reason to endorse excluded middle for identities above and beyond endorsing it for all propositions. In addition, we don't want it to turn out impossible that both $a = b$ and $a \neq b$ be true in the context of logics like **C** which can admit contradiction without embracing triviality. We also don't want to force fallacies of relevance on our friends who are concerned about that kind of thing. If we demanded that $\vdash a = b \to \big(A(a) \to A(b)\big)$, it would mean that $\vdash a = b \to (A \to A)$ for arbitrary formulae A (like those not featuring a or b) which seems a little unpalatable for logics that invalidate $\mathbf{K}'$, $B \to (A \to A)$. (See recent work by J. Michael Dunn [34] and Edwin Mares [77] for an explanation of some considerations motivated by relevant logics.)

Our conditions make the identity relation symmetric and transitive as we should hope, in the form $a = b \vdash b = a$, and $a = b, c = d \vdash a = c \leftrightarrow b = d$.

They are equivalent to adding the axiom and rule

$$x = x \qquad x = y \vdash A(x) \to A(y)$$

to our logics in the Hilbert system. In the natural deduction system identity is modelled by the axiom and rule:

$$0 \Vdash x = x \quad =\!\mathrm{I} \qquad \frac{0 \Vdash x = y}{A(x) \Vdash A(y)} \; =\!\mathrm{E}$$

The next major result makes it more clear that our characterisation of identity is the natural one.

Definition 4.9 The predicate F_0 in the language $\mathcal{L}$ where $\mu_1(0) = 2$ is a *strong identity predicate* in the model $\mathfrak{A}$ if for each $a, b \in D_{\mathfrak{A}}$, $\mathfrak{A} \vDash F_0(v_0, v_1)[a, b]$ if and only if $a = b$.

THEOREM 4.12 *If F_0 is a weak identity predicate in $\mathfrak{A}$, then there is a relational structure $\mathfrak{B}$ on a domain $D_{\mathfrak{B}} \subseteq D_{\mathfrak{A}}$, where $\mathfrak{A} \equiv \mathfrak{B}$ and in which F_0 is a strong identity predicate. In addition, if the language has no function symbols, $\mathfrak{B}$ is an elementary substructure of $\mathfrak{A}$.*

Proof: Define a relation $\approx$ on A by setting $a \approx b$ iff $\mathfrak{A} \vDash F_0(v_0, v_1)[a, b]$. By the reflexivity, symmetry and transitivity of F_0, $\approx$ is an equivalence relation. For each equivalence class, select an arbitrary member to be its representative. For each $a \in D_{\mathfrak{A}}$, let $a^{\approx}$ be the representative of the equivalence class of a. The domain $D_{\mathfrak{B}}$ of $\mathfrak{B}$ is the set of all representatives. Relations in $\mathfrak{B}$ are the restrictions of the corresponding relations in $\mathfrak{A}$ and functions in $\mathfrak{B}$ are defined by taking $s_\xi(a_1^{\approx}, \ldots, a_{\mu_2(\xi)}^{\approx})$ to be $r_\xi(a_1, \ldots, a_{\mu_2(\xi)})^{\approx}$. This makes sense by way of the easy-to-show rule to the effect that for each $b, c \in D_{\mathfrak{A}}$, if $a \approx b$ then $r_\xi(a_1, \ldots, b, \ldots, a_{\mu_2(\xi)}) \approx r_\xi(a_1, \ldots, c, \ldots, a_{\mu_2(\xi)})$. It remains to show that $\mathfrak{B}$ is elementarily equivalent to $\mathfrak{A}$.

Firstly, a bit of notation. For $d \in D_{\mathfrak{A}}^{\omega}$, the sequence $d^{\approx}$ is just $\langle d_0^{\approx}, d_1^{\approx}, \ldots \rangle$. The proof uses by two inductions. One on the complexity of terms, and the other on the complexity of formulae. Firstly, terms. We show that for each $d \in D_{\mathfrak{A}}^{\omega}$ and each term t, $\left(\mathfrak{A}^t(t)_d\right)^{\approx} = \mathfrak{B}^t(t)_{d^{\approx}}$. The base case is given by the equations

$$\left(\mathfrak{A}^t(v_n)_d\right)^{\approx} = d_n^{\approx} = \mathfrak{B}^t(v_n)_{d^{\approx}}$$

and the induction step is simple:

$$\begin{aligned}
\left(\mathfrak{A}^t\left(f_\xi(t_1,\ldots,t_{\mu_2(\xi)})\right)_d\right)^{\approx} &= r_\xi\left(\mathfrak{A}^t(t_1)_d,\ldots,\mathfrak{A}^t(t_{\mu_2(\xi)})_d\right)^{\approx} \\
&= s_\xi\left(\left(\mathfrak{A}^t(t_1)_d\right)^{\approx},\ldots,\left(\mathfrak{A}^t(t_{\mu_2(\xi)})_d\right)^{\approx}\right) \\
&= s_\xi\left(\mathfrak{B}^t(t_1)_{d^{\approx}},\ldots,\mathfrak{B}_t(t_{\mu_2(\xi)})_{d^{\approx}}\right) \\
&= \mathfrak{B}^t\left(f_\xi(t_1,\ldots,t_{\mu_2(\xi)})\right)
\end{aligned}$$

For formulae, we wish to show that for each $d \in D_{\mathfrak{A}}^{\omega}$ and each formula A, $\mathfrak{A}(A)_d = \mathfrak{B}(A)_{d^{\approx}}$. By induction on the complexity of formulae, we can proceed as follows — the base case:

$$\begin{aligned}
\mathfrak{A}\left(F_\xi(t_1,\ldots,t_{\mu_1(\xi)})\right)_d &= R_\xi\left(\mathfrak{A}^t(t_1)_d,\ldots,\mathfrak{A}^t(t_{\mu_1(\xi)})_d\right) \\
&= R_\xi\left(\mathfrak{A}^t(t_1)_d^{\approx},\ldots,\mathfrak{A}^t(t_{\mu_1(\xi)})_d^{\approx}\right) \\
&= R_\xi\left(\mathfrak{B}^t(t_1)_{d^{\approx}},\ldots,\mathfrak{B}^t(t_{\mu_1(\xi)})_{d^{\approx}}\right)
\end{aligned}$$

The cases for the propositional connectives are trivial. For the universal quantifier:

$$\begin{aligned}
\mathfrak{A}(\forall v_n A)_d = \bigcap_{a \in D_{\mathfrak{A}}} \mathfrak{A}(A)_{d(n/a)} &= \bigcap_{a \in D_{\mathfrak{A}}} \mathfrak{B}(A)_{d^{\approx}(n/a^{\approx})} \\
= \bigcap_{b \in D_{\mathfrak{B}}} \mathfrak{B}(A)_{d(n/b)^{\approx}} &= \mathfrak{B}(\forall v_n A)_{d^{\approx}}
\end{aligned}$$

and the existential quantifier case is identical. So, $\mathfrak{A} \equiv \mathfrak{B}$ as desired. If there are no function symbols it is easy to see that $\mathfrak{B} = \mathfrak{A} \restriction D_{\mathfrak{B}}$, and that we have $\mathfrak{B} \prec \mathfrak{A}$. ◁

These considerations have shown us what seems to be the semantically natural approach to identity in our favourite logics. Clearly the proof theoretical rules for identity are not as strong as one might expect; but the semantics is clearly very natural. This will come into play in Chapter 12, where we discuss identity in set theory and property theory without contraction.

For the rest of the chapter, we assume that our models come equipped with a strong identity relation. Both the predicate and the relation will be denoted by $=$, as the context will always make clear what we mean.

4.5 Going Up

Now we can make a stab at proving an upward Löwenheim-Skolem theorem.

Definition 4.10

- A relational structure is *trivial* if every sentence A is true in that relational structure. For many propositional structures $\mathfrak{P}$ (like boolean algebras of size > 1, or any of Łukasiewicz's matrices) there are no trivial relational structures. However, for structures like **BN4**, a structure with one object in the domain and every relation evaluated at b is trivial.

- A set of sentences in a language $\mathcal{L}$ is $\mathfrak{P}$-*coherent* if it has a non-trivial model in the propositional structure $\mathfrak{P}$.
- For each cardinal α, a propositional structure is α-compact if for each set of sentences Σ of size α or above, Σ is $\mathfrak{P}$-coherent iff any subset smaller than α is $\mathfrak{P}$-coherent. So any boolean algebra is $\aleph_0$-compact.
- A propositional structure is $*$-*explosive* for a unary operator $*$ just when the only models such that $\mathfrak{A} \models A \wedge *A$ for any sentence A are trivial. Logics that aren't paraconsistent (like **CK**) are $\sim$-explosive. The rest of our favourite logics are $-$-explosive, where $-A = (A \to \bot)$

THEOREM 4.13 THE UPWARD LÖWENHEIM-SKOLEM THEOREM 1 *Let Σ be a set of sentences with a non-trivial model $\mathfrak{A}$, of cardinal α, in a language $\mathcal{L}$ of size ρ and in a propositional structure $\mathfrak{P}$ that is of size δ, and is β-compact, where $\beta \leq \alpha$. Further, let $\mathfrak{P}$ be $*$-explosive for some operator $*$, such that there is a set X of cardinal β of domain elements such that $\mathfrak{A} \models *(v_0 = v_1)[a, b]$ for each $a \neq b \in X$. Then, for each γ greater than α, δ and ρ, Σ has a non-trivial model of size γ.*

Proof: Consider the extended language $\mathcal{L}'$ with new constants c_ξ $(\xi < \gamma)$. In this language, formulate the set of sentences $\Sigma \cup \Delta$ where $\Delta = \{*(c_\xi = c_\eta) : \xi \neq \eta,\ \xi, \eta < \gamma\}$. Every subset of size β of $\Sigma \cup \Delta$ can be interpreted in $\mathfrak{A}$ — with the parts of Σ interpreted exactly as before, and each of the new constants sent to appropriate members of X. By the fact that $\mathfrak{P}$ is β-compact, the whole set has a non-trivial model. Clearly this model must be at least of size γ, as the 'inequations' come out true and the logic is $*$-explosive. Then the downward theorem means we can cut this model down to one of size γ if it is too large. $\triangleleft$

We can eliminate the restriction to models $\mathfrak{A}$ in which $\mathfrak{A} \models *(v_0 = v_1)[a, b]$ for each $a \neq b \in X$, for some set X of size β in a reasonably interesting way. Firstly, take the propositional structure $\mathfrak{P}$ and extend it, if necessary, to contain maximal and minimal elements T and F. For algebras of our favourite logics this can be done conservatively, using the methods of Chapter 3. This does not alter the cardinality of the propositional structure significantly. Then take our language $\mathcal{L}$ and extend it to add a binary predicate $=_{bool}$ of *boolean identity*. Then we extend $\mathfrak{A}$ by adding the extension R of $=_{bool}$ in the obvious way.

$$R(a, b) = \begin{cases} T & \text{if } a = b, \\ F & \text{if } a \neq b. \end{cases}$$

for each $a, b \in D_{\mathfrak{A}}$. This predicate is a truly boolean identity. it is a strong identity predicate in the model $\mathfrak{A}$, although it may not be the identity relation we intend to model, it is useful for certain formal purposes, like the upward theorem.

The upward theorem applies to the model $\mathfrak{A}'$, because for every distinct pair of domain elements a, b, $\mathfrak{A} \models -(v_0 =_{bool} v_1)[a, b]$ (as the value of $v_0 =_{bool} v_1$ under this interpretation is F, and $-$ maps this value to T). The upward theorem delivers a larger model $\mathfrak{B}'$ of the desired size (as every propositional structure is $-$-explosive). Then we can ignore $=_{bool}$ and all its works in this model to give $\mathfrak{B}$, in the original language. So, we have proved

THEOREM 4.14 THE UPWARD LÖWENHEIM-SKOLEM THEOREM 2 *Let* Σ *be a set of sentences with a non-trivial model* $\mathfrak{A}$, *of cardinal* α, *in a language* $\mathcal{L}$ *of size* ρ *and in a propositional structure* $\mathfrak{P}$ *that is of size* δ, *and is* β*-compact, where* $\beta \leq \alpha$. *Then, for each* γ *greater than* α, δ *and* ρ, Σ *has a non-trivial model of size* γ.

4.6 Conclusions

We have shown that there are direct analogues to the Löwenheim-Skolem theorems from classical logic. To do this, we have had to introduce a number of new concepts, and to adapt the proofs accordingly. Perhaps the greatest benefit from this work has been the discovery of a natural-looking presentation of identity in our family of logics. This has been semantically driven, but as we will see in Chapter 12, it has some significant proof theoretic implications.

Much more work needs to be done in the area of modelling quantification in our logics. The compactness properties of arbitrary propositional structures need more study, because these limit the application of the upward theorem as it has been stated. Also, the classical results about ultraproducts need work, in order to see how results such as Łŏs's theorem generalise to our context, if in fact, they do. The classical definition of an ultraproduct relies essentially on the fact that the two-element boolean algebra is the propositional structure, and generalising it does not seem easy. So, there is plenty of work for the future. Instead of going on with that work, which takes us further from contraction-free logics, we will look at another way of modelling our logics.

Chapter 5
Simplified Semantics 1

Their philosophic spirit had ...
simplified the forms of proceeding.
— EDWARD GIBBON *The Decline and Fall of the Roman Empire* [50]

Almost any logic has an algebraic semantics — they are easy to cook up from the algebra of provably equivalent formulae of the logic. As a result, that kind of semantics does not give us a great deal of information about the logic. 'Possible worlds' semantics can give us much more. In the next few chapters we prove soundness and completeness results for the ternary relational semantics for our favourite logics and others. In this chapter, we deal with the positive fragment of the propositional logics and one way to model negation. In the next, we consider other ways to model negation. The approach taken is basically due to the work of Priest and Sylvan in their recent work "Simplified Semantics for Basic Relevant Logics" [118]. They produced a simplification of the original ternary semantics due to Routley and Meyer [128, 129, 130]. This work gave the simplification for the basic logic $\mathbf{DW}^{+}$ ($\mathbf{B}^{+}$ in their parlance) and a few negation extensions. In my paper "Simplified Semantics for Relevant Logics (and some of their rivals)" [124] I showed how their simplification can be extended to a wider class of logics. We will recount that work here, making explicit the relationship between those results and the presentations we have already seen, and showing how the semantics model each of our favourite logics (and quite a few others).

5.1 The 'd' Systems

In this work, we need to pay attention to the rules of inference warranted by a logic. We've seen two ways that a logic can define validity. The matter of interest to us at the moment is the fact that two logics can share their account of strong validity (they can share all of the same theorems) while differing about weak validity.[1] We will show some examples of logics that differ in just this way.

Definition 5.1 Given a Hilbert presentation of a logic $\mathbf{X}$, the logic $\mathbf{X}^{\mathrm{d}}$ is given by adjoining for each primitive rule from $A_1, A_2, \ldots, A_n$ to B, the rule $A_1 \vee C, A_2 \vee C, \ldots, A_n \vee C \vdash B \vee C$ which is called the *disjunctive form* of the original rule.

LEMMA 5.1 *In $\mathbf{X}^{\mathrm{d}}$, if $A_1, A_2, \ldots, A_n \vdash B$ then $A_1 \vee C, A_2 \vee C, \ldots, A_n \vee C \vdash B \vee C$. That is, in a disjunctive system, if a rule is weakly valid, its disjunctive form is also weakly valid.*

Proof: Take the proof of B from $A_1, A_2, \ldots, A_n$. Transform it by disjoining C to each sentence in the proof. As the disjunctive forms of primitive rules are also weakly valid rules in $\mathbf{X}^{\mathrm{d}}$, this is nearly a proof. All we need do is note that the axioms in the old proof are transformed to formulae of the form $A \vee C$, where A is an axiom. Simply add the formulae A and $A \to A \vee C$ somewhere earlier than $A \vee C$ in the transformed proof to ensure that $A \vee C$ is proved from axioms. $\triangleleft$

Our interest in disjunctive systems is not purely academic. The simplified semantics that we will consider will only model disjunctive systems. We'll leave the explanation of this for when we see the structures. For the moment, we'll see what this means for our logics.

THEOREM 5.2 *If the logic* $\mathbf{X}$ *has a Hilbert-style axiomatisation with only modus ponens and conjunction introduction as primitive rules, and* $\mathbf{X} \supseteq \mathbf{DW}^+ + A \to (t \to A)$, *then* $\mathbf{X}^d = \mathbf{X}$.

Proof: Clearly the disjunctive form of conjunction introduction is weakly valid in these logics, because $A \vee C, B \vee C \vdash (A \vee C) \wedge (B \vee C)$ and from the $\mathbf{DW}^+$ fact that $\vdash (A \vee C) \wedge (B \vee C) \to (A \wedge B) \vee C$ and *modus ponens*, we can derive $(A \wedge B) \vee C$.

To show that the disjunctive form of *modus ponens* is weakly valid, we proceed as follows. In $\mathbf{DW} \vdash (A \to B) \wedge (C \to C) \to (A \vee C \to B \vee C)$, and so $\vdash (A \to B) \wedge t \to ((A \vee C) \wedge t \to B \vee C)$.

From $\vdash C \to (t \to C)$ we can prove that $\vdash C \to ((A \vee C) \wedge t \to B \vee C)$ so by basic $\mathbf{DW}^+$ twiddling we have

$$\vdash ((A \to B) \vee C) \wedge t \to ((A \vee C) \wedge t \to B \vee C)$$

which gives us the weak validity of the disjunctive form of *modus ponens*. ◁

COROLLARY $\mathbf{C} = \mathbf{C}^d$, $\mathbf{CK} = \mathbf{CK}^d$, $\mathbf{LI}^+ = \mathbf{LI}^{+\,d}$, *and* $\mathbf{R} = \mathbf{R}^d$.

THEOREM 5.3 *If the logic* $\mathbf{X}^d$ *is prime, then the theorems of* $\mathbf{X}$ *are identical to the theorems of* $\mathbf{X}^d$.

Proof: Simple. If $\vdash A$ in $\mathbf{X}^d$ then A has a proof all of the sentences in which are theorems. In this proof, if we apply the disjunctive form of a rule to premises of the form $A_1 \vee C, \ldots, A_n \vee C$ to get $B \vee C$, then by the primeness of $\mathbf{X}^d$, either C is a theorem or $A_1, \ldots, A_n$ are theorems. In the first case, $B \vee C$ follows from C by the usual steps; and in the second, we use the non-disjunctive form of the original rule to derive B, and then $B \vee C$ as before. ◁

LEMMA 5.4 *If* $\mathbf{X}$ *is proved to be prime by means of the metacompleteness argument of Chapter 2, then* $\mathbf{X}^d$ *is prime too.*

Proof: If a proof works for $\mathbf{X}$ then the only changes for $\mathbf{X}^d$ are in the clause that shows that if $\vdash A$ then $\mathcal{T}(A)$. For this, we need to show that the disjunctive form of any rule is $\mathcal{T}$-preserving too. Assume that the rule from A_i (for $i = 1$ to n) to B is $\mathcal{T}$-preserving. If $\mathcal{T}(A_i \vee C)$ for each i, then either $\mathcal{T}(C)$ (and hence $\mathcal{T}(B \vee C)$) or $\mathcal{T}(A_i)$ for each i, and hence $\mathcal{T}(B)$ (giving $\mathcal{T}(B \vee C)$). So, the disjunctive form is $\mathcal{T}$-preserving too. ◁

COROLLARY *The theorems of* $\mathbf{DW}$, $\mathbf{TW}$, $\mathbf{L}^+$ *and* $\mathbf{EW}$ *are identical to those of the corresponding disjunctive systems.*

This comes apart in contraction-added logics like $\mathbf{E}$. Brady noted this result [16].

THEOREM 5.5 $\sim A \vee ((A \to B) \to B)$ *is a theorem of* $\mathbf{E}^d$, *but not of* $\mathbf{E}$.

Proof: For $\mathbf{E}^d$ note that $\vdash \sim A \vee A$ and the disjunctive form of restricted assertion shows that $\sim A \vee A \vdash \sim A \vee ((A \to B) \to B)$. For $\mathbf{E}$, `MaGIC` assures me that there is a four element $\mathbf{E}$ps that falsifies $\sim A \vee ((A \to B) \to B)$. The reader is invited to try his or her hand at finding it. $\triangleleft$

5.2 Semantics for DW$^+$

Priest and Sylvan's main construction is their semantics for $\mathbf{DW}^+$. Their semantics is a simplified version of the original ternary relational semantics for relevant logics. The important definitions concerning the structure are collected here.

Definition 5.2 An *interpretation* for the language is a quadruple $\langle g, W, R, \models\rangle$, where W is a set of indices, $g \in W$ is the base index, R is a ternary relation on W, and $\models$ is a relation defined firstly between indices and atomic formulae, and then assigned to all formulae inductively as follows:

- $w \models A \wedge B \iff w \models A$ and $w \models B$,
- $w \models A \vee B \iff w \models A$ or $w \models B$,
- $g \models A \to B \iff$ for all $x \in W$ $(x \models A \Rightarrow x \models B)$,

and for $x \neq g$,

- $x \models A \to B \iff$ for all $y, z \in W$ $(Ryxz \Rightarrow (y \models A \Rightarrow z \models B))^2$.

Then weak validity in the model is defined in terms of truth preservation at g, the base index. In other words we have:

Definition 5.3 $\Theta \vDash A$ iff for all interpretations $\langle g, W, R, \models\rangle$ and for each $B \in \Theta$, if $g \models B$ then $g \models A$.

EXCURSUS: Note that $\vDash$ and $\models$ are two completely different things. The relation $\vDash$ is the model-theoretic analogue of the relation $\vdash$. The relation $\models$ is internal to models, and quite different. We will use these two very similar notations for different notions because their use is entrenched, and because we will very rarely use '$\vDash$'. $\square$

This explains the why logics modelled by this scheme are always disjunctive systems. If $\Theta \vDash A$, then consider Θ' given by disjoining each element of Θ with C. Then if in an interpretation for each $B \vee C \in \Theta'$, $g \models B \vee C$, we have two alternatives. Firstly, $g \models C$, in which case $g \models A \vee C$. Otherwise, $g \models B$ for each $B \vee C \in \Theta'$, or equivalently, for each $B \in \Theta$. This means that $g \models A$, and hence $g \models A \vee C$. So, either way, $g \models A \vee C$. This means that if $\Theta \vDash A$, then $\Theta' \vdash A \vee C$. If a rule is weakly valid, so is its disjunctive form. It follows immediately from the definition of weak validity in terms of truth preservation at the base index.

The soundness and completeness result for $\mathbf{DW}^+$can then be concisely stated as follows.

THEOREM 5.6 *If $\Theta \cup \{A\}$ is a set of sentences, then*

$$\Theta \vdash A \iff \Theta \vDash A,$$

where $\vdash$ is weak validity in $\mathbf{DW}^+$.

In this chapter we will make a cosmetic alteration to the above definition of an interpretation. It is easily seen that the truth conditions for '$\rightarrow$' can be made univocal if we define R to satisfy $Rxgy \iff x = y$. From now, we will use the following definition of an interpretation.

Definition 5.4 An *interpretation* for the language is now a quadruple $\langle g, W, R, \models \rangle$, where W is a set of indices, $g \in W$ is the base index, R is a ternary relation on W satisfying $Rxgy \iff x = y$, and $\models$ is a relation defined firstly on $W \times \mathrm{AProp}$ and then assigned to all formulae inductively as follows:

- $w \models A \wedge B \iff w \models A$ and $w \models B$,
- $w \models A \vee B \iff w \models A$ or $w \models B$,
- $x \models A \rightarrow B \iff$ for all $y, z \in W$ $(Ryxz \Rightarrow (y \models A \Rightarrow z \models B))$.

  And to model fusion we add the clause:
- $x \models A \circ B \iff$ for some $y, z \in W$ $(Ryzx,\ y \models A$ and $z \models B)$.

Semantic consequence is still defined in terms of truth preservation at g, the base index. It is clear how one can translate between the two notions of an interpretation. The reason we use this altered definition is in the phrasing of conditions on R which give extensions of the logic $\mathbf{DW}^+$. It is much less tedious to write '$Rabc \Rightarrow Rbac$' than it is to write '$Rabc \Rightarrow Rbac$ for $a \neq g$ and $Rgbb$ for each b, but these are equivalent definitions under each variety of interpretation.

Excursus: Priest and Sylvan do not give the modelling condition for fusion in their paper. Its presence makes the proofs no more difficult. □

5.3 Soundness

The soundness theorem for $\mathbf{DW}^+$ and its extensions is simple (but tedious) to prove. We state it and prove it in part, leaving the rest to the interested reader.

Theorem 5.7 *For each row in the table below, the logic $\mathbf{DW}^+$ with the axiom (or rule) added is sound with respect to the class of $\mathbf{DW}^+$ interpretations $\langle g, W, R, \models \rangle$ where R satisfies the corresponding condition.*

PMP	$A \wedge (A \rightarrow B) \rightarrow B$	$Raaa$
CSyll	$(A \rightarrow B) \wedge (B \rightarrow C) \rightarrow (A \rightarrow C)$	$Rabc \Rightarrow R^2(ab)bc$
B	$(A \rightarrow B) \rightarrow ((C \rightarrow A) \rightarrow (C \rightarrow B))$	$R^2a(bc)d \Rightarrow R^2(ab)cd$
B′	$(A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))$	$R^2a(bc)d \Rightarrow R^2(ac)bd$
W	$(A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B)$	$Rabc \Rightarrow R^2a(ab)c$
CI	$A \rightarrow ((A \rightarrow B) \rightarrow B)$	$Rabc \Rightarrow Rbac$
C	$(A \rightarrow (B \rightarrow C)) \rightarrow (B \rightarrow (A \rightarrow C))$	$R^2a(bc)d \Rightarrow R^2b(ac)d$
S	$(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$	$R^2a(bc)d \Rightarrow R^3(ab)(ac)d$
CS	$(A \rightarrow B) \rightarrow ((A \rightarrow (B \rightarrow C)) \rightarrow (A \rightarrow C))$	$R^2a(bc)d \Rightarrow R^3(ac)(ab)d$
C″	$\dfrac{A}{(A \rightarrow B) \rightarrow B}$ and its disjunctive forms	$Rgaa$

Table 5.1

Where we have defined: $R^2(ab)cd = (\exists x)(Rabx \wedge Rxcd)$, $R^2a(bc)d = (\exists x)(Rbcx \wedge Raxd)$ *and* $R^3(ab)(cd)e = (\exists x)(R^2abxe \wedge Rcdx)$.

Proof: To prove this, we take an arbitrary $\mathbf{DW}^+$ interpretation $\langle g, W, R, \models\rangle$ and assume that the relation R satisfies one of the conditions. Then it suffices to demonstrate that any instance of the corresponding axiom is supported at g. (Or that the corresponding rules are always truth preserving at g.) The soundness results are simple and mechanical.

For those who like a quick proof, it suffices to note that any $\mathbf{DW}^+$ interpretation $\langle g, W, R, \models\rangle$ is also an old-style interpretation — with the proviso that $Rxgy \iff x = y$. Hence, we can shamelessly appropriate the soundness proofs from Routley's work using the unsimplified semantics [134], pages 304–305. The only thing necessary to add to this proof are the rules for fusion. We will show that $g \models A \circ B \to C$ if and only if $g \models B \to (A \to C)$. Firstly, suppose that $g \models A \circ B \to C$ and that $g \not\models B \to (A \to C)$. Then for some x, $x \models B$ and $x \not\models A \to C$. This latter fact means that for some y, z where $Ryxz$, $y \models A$ and $z \not\models C$. However, $Ryxz$, $y \models A$ and $x \models B$ give $z \models B \circ C$ and this together with $g \models A \circ B \to C$ gives $z \models C$, contradicting what we have already seen.

This style of argument is very common in soundness proofs, and in using the semantics for checking validity. Instead of using twisted prose like that to explain why the requisite models cannot be found, we use a standard tableaux formalism. The proof then goes like this.

1	$g \models A \circ B \to C$	
2	$g \not\models B \to (A \to C)$	
3	$x \models B$	[2]
4	$x \not\models A \to C$	[2]
5	$Ryxz$	[4]
6	$y \models A$	[4]
7	$z \not\models C$	[4]
8	$z \models A \circ B$	[3,5,6]
9	$z \models C$	[1,8]
10	$\times$	[7,9]

As usual in tableaux systems, we start by assuming what we wish to disprove — here by writing it on two lines — that $g \models A \circ B \to C$ and $g \not\models B \to (A \to C)$. Then each line is justified by the previous lines mentioned in the brackets on the right. These all follow from the evaluation clauses of the semantics. We need just be careful that when interpreting existential claims (there exists a world such that ...) the world we introduce is new to the branch. We have introduced worlds at lines 3 and 4 (from line 2) and 5, 6 and 7 (from line 4). Then closure occurs when the resulting system of worlds contradicts a modelling condition — this shows that the assumptions made at the start cannot hold.

Now we will show that the converse fusion rule holds too, using the tableaux system.

1	$g \models B \to (A \to C)$	
2	$g \not\models A \circ B \to C$	
3	$x \models A \circ B$	[2]
4	$x \not\models C$	[2]
5	$Ryzx$	[3]
6	$y \models A$	[3]
7	$z \models B$	[3]
8	$z \models A \to C$	[1,7]
9	$x \models C$	[5,6,8]
10	$\times$	[4,9]

We can work the proofs of the other axioms independently too. We give two examples to show how they go

PMP. Suppose that $Raaa$ for each $a \in W$. Then $g \models A \wedge (A \to B) \to B$.

1	$g \not\models A \wedge (A \to B) \to B$	
2	$x \models A \wedge (A \to B)$	[1]
3	$x \not\models B$	[1]
4	$x \models A$	[2]
5	$x \models A \to B$	[2]
6	$Rxxx$	[PMP]
7	$x \models B$	[4,5,6]
8	$\times$	[4,7]

C″. Assume that $Rgaa$ for each a. Then $g \models A \vee C$ only if $g \models ((A \to B) \to B) \vee C$.

1	$g \models A \vee C$	
2	$g \not\models ((A \to B) \to B) \vee C$	
3	$g \not\models (A \to B) \to B$	[2]
4	$g \not\models C$	[2]
5	$x \models A \to B$	[3]
6	$x \not\models B$	[3]
7	$g \models A$	[1,4]
8	$Rgxx$	[C”]
9	$x \models B$	[5,7,8]
10	$\times$	[6,9]

◁

5.4 Results Concerning Prime Theories

The completeness result for $\mathbf{DW}^+$ relies on a standard model construction, where the indices are prime theories. We will need certain definitions and facts about prime theories to prove this result for logics extending $\mathbf{DW}^+$.

Definition 5.5

- $\Sigma \vdash_\Pi A \iff \Sigma \cup \Pi \vdash A$.
- Σ is a Π-*theory* $\iff$
  - $A, B \in \Sigma \Rightarrow A \wedge B \in \Sigma$,
  - $\vdash_\Pi A \to B \Rightarrow (A \in \Sigma \Rightarrow B \in \Sigma)$.
- Σ is *prime* $\iff (A \vee B \in \Sigma \Rightarrow A \in \Sigma$ or $B \in \Sigma)$.
- If X is any set of sets of formulae, the ternary relation R on X is defined thus:

$$R\Sigma\Gamma\Delta \iff (C \to D \in \Gamma \Rightarrow (C \in \Sigma \Rightarrow D \in \Delta)).$$

- $\Sigma \vdash_\Pi \Delta \iff$ for some $D_1, \ldots, D_n \in \Delta$, we have $\Sigma \vdash_\Pi D_1 \vee \ldots \vee D_n$.
- $\vdash_\Pi \Sigma \to \Delta \iff$ for some $S_1, \ldots, S_n \in \Sigma$ and $D_1, \ldots, D_n \in \Delta$ we have $\vdash_\Pi S_1 \wedge \ldots \wedge S_n \to D_1 \vee \ldots \vee D_n$.
- Σ is Π-*deductively closed* $\iff (\Sigma \vdash_\Pi A \Rightarrow A \in \Sigma)$.
- Where L is the set of all $\mathcal{L}$-sentences, $\langle \Sigma, \Delta \rangle$ is a Π-*partition* if and only if :
  - $\Sigma \cup \Delta = L$,
  - $\not\vdash_\Pi \Sigma \to \Delta$.

EXCURSUS: Priest and Sylvan define $\vdash_\Pi$ slightly differently. They take it that $\Sigma \vdash_\Pi A$ just when $\Sigma \cup \Pi_\to \vdash A$ where $\Pi_\to$ is the set of all conditionals in Π. We need slightly more for this in order to model t (which we meet in Section 5.6). The different definition makes no difference to any of the proofs. □

In all of the above definitions, if Π is the empty set, the prefix 'Π-' is omitted; so a $\varnothing$-theory is simply a theory, and so on. The following results are proved by Priest and Sylvan, so the proofs will not be repeated here.

LEMMA 5.8

- *If $\langle \Sigma, \Delta \rangle$ is a Π-partition then Σ is a prime Π-theory.*
- *If $\not\vdash_\Pi \Sigma \to \Delta$ then there are $\Sigma' \supseteq \Sigma$ and $\Delta' \supseteq \Delta$ such that $\langle \Sigma', \Delta' \rangle$ is a Π-partition.*
- *If Σ is a Π-theory, and Δ is closed under disjunction, with $\Sigma \cap \Delta = \varnothing$, then there is a $\Sigma' \supseteq \Sigma$ such that $\Sigma' \cap \Delta = \varnothing$ and Σ' is a prime Π-theory.*
- *If Π is a prime Π-theory, is Π-deductively closed, and $A \to B \notin \Pi$, then there is a prime Π-theory, Γ, such that $A \in \Gamma$ and $B \notin \Gamma$.*
- *If Σ is a prime Π-theory with $C \to D \notin \Sigma$, then there are prime Π-theories, Γ and Δ such that $R\Sigma\Gamma\Delta$, $C \in \Gamma$ and $D \notin \Delta$.*

Given some Π-theory Σ with various properties, we are interested in finding a *prime* Π-theory $\Sigma' \supseteq \Sigma$ that retains those properties, because our canonical model structures require the indices to be prime. The following lemmas do this, and hence they are called *priming lemmas*. The content of these is contained in either Priest and Sylvan's paper [118] or Routley's book [134], but not in the form we need; we repeat them here for completeness' sake.

LEMMA 5.9 *If Σ, Γ and Δ are Π-theories, such that $R\Sigma\Gamma\Delta$ and Δ is prime, then there is a prime $\Sigma' \supseteq \Sigma$ where $R\Sigma'\Gamma\Delta$.*

Proof: We construct Σ' by defining $\Theta = \{A : (\exists B \notin \Delta)(A \to B \in \Gamma)\}$. Θ is closed under disjunction, for suppose $A_1, A_2 \in \Theta$. Then there are $B_1, B_2 \notin \Delta$ such that $A_1 \to B_1, A_2 \to B_2 \in \Gamma$. As Δ is prime, $B_1 \vee B_2 \notin \Delta$. It is easy to show that $A_1 \vee A_2 \to B_1 \vee B_2 \in \Gamma$, as Σ is a Π-theory. So we have $A_1 \vee A_2 \in \Theta$. Moreover, $\Sigma \cap \Theta = \varnothing$. For, suppose that $A \in \Sigma \cap \Theta$. Then there is some $B \notin \Delta$ where $A \to B \in \Gamma$, contradicting $R\Sigma\Gamma\Delta$.

Thus, applying a part of Lemma 5.8 there is a prime Π-theory $\Sigma' \supseteq \Sigma$ where $\Sigma' \cap \Theta = \varnothing$. To see that $R\Sigma'\Gamma\Delta$, let $A \to B \in \Gamma$ and $A \in \Sigma'$. Then $A \notin \Theta$, so we have $B \in \Delta$. ◁

Lemma 5.10 *If Σ, Γ and Δ are Π-theories, such that $R\Sigma\Gamma\Delta$ and Δ is prime, then there is a prime $\Gamma' \supseteq \Gamma$ where $R\Sigma\Gamma'\Delta$.*

Proof: This time let $\Theta = \{A : (\exists B, C) \text{ where } \vdash_\Pi A \to (B \to C), B \in \Sigma \text{ and } C \notin \Delta\}$. Θ is closed under disjunction, for if $A_1, A_2 \in \Theta$, then for some $B_1, B_2 \in \Sigma$ and $C_1, C_2 \notin \Delta$, $\vdash_\Pi A_1 \to (B_1 \to C_1)$ and $\vdash_\Pi A_2 \to (B_2 \to C_2)$. Hence $B_1 \wedge B_2 \in \Sigma$ (as Σ is a Π-theory) and $C_1 \vee C_2 \notin \Delta$ (as Δ is prime). It is straightforward to show that then $\vdash_\Pi A_1 \vee A_2 \to (B_1 \wedge B_2 \to C_1 \vee C_2)$, because we have $\vdash_\Pi A_1 \vee A_2 \to ((B_1 \to C_1) \vee (B_2 \to C_2))$. So we have $A_1 \vee A_2 \in \Theta$.

$\Gamma \cap \Theta = \varnothing$. For if $D \in \Gamma \cap \Theta$, then there are $B \in \Sigma$ and $C \notin \Delta$ where $\vdash_\Pi D \to (B \to C)$. Then as $D \in \Gamma$, and as Γ is a Π-theory we have $B \to C \in \Gamma$, contradicting $R\Sigma\Gamma\Delta$.

Lemma 5.8 then gives a prime $\Gamma' \supseteq \Gamma$, disjoint from Θ. $R\Sigma'\Gamma\Delta$ obtains, as if $B \to C \in \Gamma$, $B \to C \notin \Theta$, so for all B' and C' where $\vdash_\Pi (B \to C) \to (B' \to C')$, if $B' \in \Sigma$, then $C' \in \Delta$. But $\vdash_\Pi (B \to C) \to (B \to C)$, so we have our result. ◁

Lemma 5.11 *If Σ, Γ and Δ are Π-theories, such that $R\Sigma\Gamma\Delta$ and Δ is prime, then there are prime $\Sigma' \supseteq \Sigma$ and $\Gamma' \supseteq \Gamma$ where $R\Sigma'\Gamma'\Delta$.*

Proof: Apply the previous two lemmas. ◁

Lemma 5.12 *If Σ, Γ and Δ are Π-theories, such that $R\Sigma\Gamma\Delta$ and $D \notin \Delta$, then there are prime Π-theories Σ' and Δ' such that $R\Sigma'\Gamma\Delta'$, $\Sigma \subseteq \Sigma'$, $D \notin \Delta'$, and $\Delta \subseteq \Delta'$.*

Proof: First construct Δ'. Take Θ_1 to be the closure of $\{D\}$ under disjunction. As $\vdash_\Pi D \vee \ldots \vee D \to D$ and Δ is a Π-theory, $\Delta \cap \Theta_1 = \varnothing$. By a part of Lemma 5.8, there is a prime Π-theory $\Delta' \supseteq \Delta$ where $\Delta' \cap \Theta_1 = \varnothing$, and it follows that $R\Sigma\Gamma\Delta'$, and that $D \notin \Delta'$.

To construct Σ', take Θ_2 to be $\{A : \exists B \notin \Delta' \text{ where } A \to B \in \Gamma\}$. Θ_2 is closed under disjunction and $\Sigma \cap \Theta_2 = \varnothing$. (See Priest and Sylvan [118] for a proof, or take it as an exercise.) Lemma 5.8 shows that there is a prime Π-theory $\Sigma' \supseteq \Sigma$ where $\Sigma' \cap \Theta_2 = \varnothing$. To show that $R\Sigma'\Gamma\Delta'$, take $A \to B \in \Gamma$ and $A \in \Sigma'$. Then $A \notin \Theta_2$, and hence $B \in \Delta'$. ◁

This gives us the completeness of the semantics for $\mathbf{DW}^+$. Given a set of formulae $\Theta \cup \{A\}$, such that $\Theta \nvdash A$, we construct an interpretation in which Θ holds at the base index, but A doesn't. Firstly, note that there is a prime theory Π such that $\Pi \supseteq \Theta$, but $A \notin \Pi$, by Lemma 5.8. The indices of the interpretation are the Π-theories, g is Π itself and R is as defined above, except that $R\Gamma\Pi\Delta$ if and only if $\Gamma = \Delta$.[3] Then we determine $\models$, by assigning $\Sigma \models p \iff p \in \Sigma$ for each propositional variable p and Π-theory Σ. It can then be proved that $B \models \Sigma \iff B \in \Sigma$ for each formula B, so we have that Θ holds at Π, the base index, and A does not.

5.5 Completeness

To show completeness for the extensions, it is usual to show that any canonical model of the logic in question satisfies the conditions corresponding to the logic. So for **PMP**, we show that any canonical model formed from the logic $\mathbf{DW}^+$+**PMP** satisfies the condition $Raaa$ for each a. The completeness result then follows immediately as described in the last section.

Unfortunately for us, the results of this form appear to break down when we extend the logic too far beyond $\mathbf{DW}^+$. A simple example is given by the logic $\mathbf{DW}^+$+**W**. It is not at all clear that the canonical interpretation of this logic satisfies the condition $Rabc \Rightarrow R^2a(ab)c$. The reason is as follows: Suppose that Σ, Γ and Δ are prime Π-theories such that $R\Sigma\Gamma\Delta$. We wish to find a prime Π-theory Ω such that $R\Sigma\Gamma\Omega$ and $R\Gamma\Omega\Delta$. The general approach is to let Ω be the smallest set satisfying the first condition — it will turn out to be a Π-theory, and a priming lemma gives us a corresponding prime theory Ω' — and then we demonstrate that Ω satisfies the second condition. (And a priming lemma ensures that Ω' will also. The details are given when we get to the proof. They are sketched here to motivate what follows.) This proof goes through, except for the case when Ω turns out to be Π. In that case $R\Gamma\Omega\Delta$ if and only if $\Gamma = \Delta$, and this does not seem to follow from what we have assumed. It is at this step that many of the completeness arguments fail.

So instead of using the original canonical interpretation, we will use another, in which the standard arguments work. We note that the difficulty with the standard argument arises when a Π-theory (say Ω) constructed to satisfy $R\Gamma\Omega\Delta$, turns out to be Π itself. What would solve the problem is some *other* Π-theory which has exactly the same truths as Π, but which has 'orthodox' R-relations with other Π-theories. In other words, we wish to have a Π-theory Π', which satisfies $R\Gamma\Pi'\Delta$ if and only if $A \to B \in \Pi' \Rightarrow (A \in \Gamma \Rightarrow B \in \Delta)$, instead of the more restrictive condition of $\Gamma = \Delta$. Then this index will take the place of Π, whenever we need it in the second place of an R-relation. This is only by way of motivation, and does not constitute a proof. We will formally explicate this model structure, and prove the completeness theorems with it.

Definition 5.6 Given that Π is a prime Π-theory of a logic $\mathbf{X}$ extending $\mathbf{DW}^+$, an *almost-canonical interpretation* for $\mathbf{X}$ is a 4-tuple $\langle\langle\Pi, 1\rangle, W, R, \models\rangle$, where

- $W = \{\langle\Sigma, 0\rangle : \Sigma \text{ is a prime } \Pi\text{-theory}\} \cup \{\langle\Pi, 1\rangle\}$,
- R is defined on W^3 to satisfy:
  - $Rx\langle\Pi, 1\rangle y$ if and only if $x = y$,
  - $R\langle\Gamma, i\rangle\langle\Sigma, 0\rangle\langle\Delta, j\rangle$ if and only if for each A and B, $A \to B \in \Sigma \Rightarrow (A \in \Gamma \Rightarrow B \in \Delta)$, for $i, j \in \{0, 1\}$,
- $\langle\Sigma, i\rangle \models A$ if and only if $A \in \Sigma$, for $i \in \{0, 1\}$.

In a moment, we will demonstrate that this actually is an interpretation (by showing that $\models$ satisfies the inductive properties needed for an interpretation), but first, we will simplify our notation. It is clear that the almost-canonical interpretation is simply the canonical interpretation with another index with the same truths as the base index, but entering into different R-relations (when it appears in the second place of R). Other than that it is identical, so we will ignore the ordered-pair notation, and simply write Π for what was $\langle \Pi, 1 \rangle$, the base index; Π' for $\langle \Pi, 0 \rangle$, its double; and for each other index $\langle \Sigma, 0 \rangle$, we will simply use Σ. Further, instead of writing $A \in \langle \Sigma, i \rangle$, we simply write $A \in \Sigma$. In this way, we cut down on notation, and the parallel with the canonical interpretation is made clear. In fact, you can ignore the whole business with ordered-pairs, and simply imagine W to be the set of all prime Π-theories, each of which is painted blue, and a single set with the same elements as Π, which is painted red. The red one is the base index, and has R defined on it in its own peculiar way, and the blue ones have R defined on them as normal. However you think of it, seeing it in use will (hopefully) make it clear. The first proof demonstrates that it is actually an interpretation.

THEOREM 5.13 *The almost-canonical interpretation is worthy of its name; that is, it is an interpretation.*

Proof: Define $\models$ by requiring that:

$$\Sigma \models p \text{ if and only if } p \in \Sigma, \text{ for } p \text{ a propositional parameter,}$$

and that it satisfy the usual inductive definitions of an interpretation. We simply need to show that $\Sigma \models A$ if and only if $A \in \Sigma$ for *every* formula A. We do this by induction on the complexity of the formulae.

- It works by stipulation on the base case.
- $\Sigma \models A \wedge B$ if and only if $\Sigma \models A$ and $\Sigma \models B$ (by the inductive definition of $\models$), if and only if $A \in \Sigma$ and $B \in \Sigma$ (by the inductive hypothesis), if and only if $A \wedge B \in \Sigma$ (as Σ is a Π-theory).
- $\Sigma \models A \vee B$ if and only if $\Sigma \models A$ or $\Sigma \models B$ (by the inductive definition of $\models$), if and only if $A \in \Sigma$ or $B \in \Sigma$ (by the inductive hypothesis), if and only if $A \vee B \in \Sigma$ (as Σ is a *prime* Π-theory).
- $\Sigma \models A \to B$ if and only if for each Γ, Δ where $R\Gamma\Sigma\Delta$ ($\Gamma \models A \Rightarrow \Delta \models B$), if and only if for each Γ, Δ where $R\Gamma\Sigma\Delta$ ($A \in \Gamma \Rightarrow B \in \Delta$).

  We desire to show that this last condition obtains if and only if $A \to B \in \Sigma$. We take this in two cases — firstly when Σ is the base index.

  Then, $\Pi \models A \to B$ if and only if for each Γ, ($A \in \Gamma \Rightarrow B \in \Gamma$), if and only if $A \to B \in \Pi$, as Γ is a Π-theory, and by the fourth part of Lemma 5.8.

  If Σ is not the base index, then $R\Gamma\Sigma\Delta$ if and only if $R'\Gamma\Sigma\Delta$, where R' is the relation on Π-theories defined univocally as $R'\Gamma\Sigma\Delta$ if and only if $(\forall A \to B \in \Sigma)(A \in \Gamma \Rightarrow B \in \Delta)$. Then we have: $\Sigma \models A \to B$, if and only if for each Γ, Δ where $R\Gamma\Sigma\Delta$, $(A \in \Gamma \Rightarrow B \in \Delta)$, if and only if $A \to B \in \Sigma$, by the definition of R, and by the fifth part of Lemma 5.8.

- $\Sigma \models A \circ B$ if and only if for some Γ, Δ where $R\Gamma\Delta\Sigma$, $A \in \Gamma$ and $B \in \Delta$. Now if this holds, $\vdash_\Pi B \to (A \to A \circ B)$ gives $A \to A \circ B \in \Delta$ and hence, $A \circ B \in \Sigma$, as desired. Conversely, if $A \circ B \in \Sigma$, we construct suitable Γ and Δ as follows. Let $\Gamma' = \{A' : \vdash_\Pi A \to A'\}$ and $\Delta' = \{B' : \vdash_\Pi B \to B'\}$. If $C \to D \in \Delta'$ and $C \in \Gamma'$, we have $\vdash_\Pi B \to (C \to D)$ and $\vdash_\Pi A \in C$, so $\vdash_\Pi C \circ B \to D$ and $\vdash_\Pi A \circ B \to C \circ B$ give $D \in \Sigma$. So, unless $\Delta' = \Pi$, $R\Gamma'\Delta'\Sigma$, and an application of Lemma 5.11 produces prime Π-theories $\Delta \supseteq \Delta'$, $\Gamma \supseteq \Gamma'$ such that $R\Gamma\Delta\Sigma$. If $\Delta' = \Pi$, then $\vdash_\Pi t \leftrightarrow B$. Then $A \in \Sigma$ (as $\vdash_\Pi A \circ B \leftrightarrow A$) and $R\Sigma\Pi\Sigma$ gives us the result.

So for *any* index Σ, $\Sigma \models A \to B$ if and only if $A \to B \in \Sigma$. This completes the inductive proof. ◁

We now have enough results to prove completeness.

THEOREM 5.14 *For each row in Table 5.1 the logic* $\mathbf{DW}^+$ *with the axiom (or rule) added is complete with respect to the class of* $\mathbf{DW}^+$ *interpretations* $\langle g, W, R, \models\rangle$ *where* R *satisfies the corresponding condition.*

Proof: We will demonstrate these individually, showing that the canonical model of any logic satisfying $\mathbf{DW}^+$ and an axiom (or rule) must satisfy the corresponding condition.

PMP. We wish to show that $R\Sigma\Sigma\Sigma$ for each prime Π-theory Σ, under the assumption of **PMP**. Consider $A \to B \in \Sigma$, and $A \in \Sigma$. Thus $A \wedge (A \to B) \in \Sigma$, and $\vdash_\Pi A \wedge (A \to B) \to B$ gives $B \in \Sigma$, and so, if $\Sigma \neq \Pi$, $R\Sigma\Sigma\Sigma$. If $\Sigma = \Pi$, the result follows immediately.

CSyll. We wish to show that for all prime Π-theories Σ, Γ and Δ where $R\Sigma\Gamma\Delta$, there is a prime Π-theory Θ' where $R\Sigma\Gamma\Theta'$ and $R\Theta'\Gamma\Delta$. Let $\Theta = \{B : (\exists A)(A \to B \in \Gamma) \wedge (A \in \Sigma)\}$. Θ is a Π-theory because:

- $B_1, B_2 \in \Theta$ means that there are $A_1, A_2 \in \Sigma$ where $A_1 \to B_1, A_2 \to B_2 \in \Gamma$. So, $A_1 \wedge A_2 \in \Sigma$ and $(A_1 \to B_1) \wedge (A_2 \to B_2) \in \Gamma$. But this gives $A_1 \wedge A_2 \to B_1 \wedge B_2 \in \Gamma$. (Because $\vdash_\Pi (A_1 \to B_1) \wedge (A_2 \to B_2) \to (A_1 \wedge A_2 \to B_1 \wedge B_2)$.) This ensures that $B_1 \wedge B_2 \in \Theta$.
- If $\vdash_\Pi A \to B$ and $A \in \Theta$, there is a $C \in \Sigma$ where $C \to A \in \Gamma$, and because we have $\vdash_\Pi (C \to A) \to (C \to B)$ by prefixing, we then have $C \to B \in \Gamma$. This gives $B \in \Theta$, as desired.

Now $R\Sigma\Gamma\Theta$ is true by definition (even if $\Sigma = \Pi$, in which case $\Gamma = \Delta = \Theta$, and we have our result immediately). To show that $R\Theta\Gamma\Delta$, if $A \to B \in \Gamma$ and $A \in \Theta$, there is a $C \in \Sigma$ where $C \to A \in \Gamma$, so $(C \to A) \wedge (A \to B) \in \Gamma$. **CSyll** ensures that $C \to B \in \Gamma$, and hence $B \in \Delta$ as $R\Sigma\Gamma\Delta$.

By Lemma 5.9 there is a prime $\Theta' \supseteq \Theta$ where $R\Theta'\Gamma\Delta$, and $R\Sigma\Gamma\Theta'$ is ensured by $\Theta' \supseteq \Theta$. If as sets $\Theta' = \Pi$, then we select it can be either Π or Π' for this case — in other cases the choice is important. Whatever we take Θ' to be, we have our result.

Before we go on to the next case, we would do well to note some features of this one. Θ, as we defined it, is the smallest set satisfying $R\Sigma\Gamma\Theta$, and fortunately for us, it is a Π-theory. We will use this construction often, and we will not rewrite the proof that the set so formed is a Π-theory.

$\mathbf{B}'$. Assume that $\mathbf{B}'$ holds, and consider arbitrary prime Π-theories Σ, Θ, Γ, Δ and Ξ, where $R\Gamma\Theta\Xi$ and $R\Theta\Xi\Delta$. We wish to find a prime Π-theory Ω' where $R\Sigma\Theta\Omega'$ and $R\Omega'\Gamma\Delta$. To this end, let $\Omega = \{B : (\exists A)(A \to B \in \Theta) \wedge (A \in \Sigma)\}$; this is a Π-theory as before. $R\Sigma\Theta\Omega$ is immediate (even when $\Theta = \Pi$, in which case $\Gamma = \Xi$, and $\Sigma = \Omega$, yielding the result). To show that $R\Omega\Gamma\Delta$, let $A \to B \in \Gamma$ and $A \in \Omega$. Then there is a $C \in \Sigma$ where $C \to A \in \Theta$, and as $\vdash_\Pi (C \to A) \to ((A \to B) \to (C \to B))$, we have $(A \to B) \to (C \to B) \in \Theta$. $R\Gamma\Theta\Xi$ then ensures that $C \to B \in \Xi$, and $R\Sigma\Xi\Delta$ then gives $B \in \Delta$. This means that $R\Omega\Gamma\Delta$, if $\Gamma \neq \Pi$. In this case a priming lemma then ensures the existence of a prime $\Omega' \supseteq \Omega$ where $R\Omega'\Gamma\Delta$ and $R\Sigma\Theta\Omega'$, and hence our result. Again, if as sets $\Omega' = \Pi$, then it is unimportant whether we take Ω' to be Π or Π'. (For other cases, if it is unimportant, we will fail to mention that fact.)

If, on the other hand $\Gamma = \Pi$, set $\Omega' = \Delta$. Then $R\Sigma\Theta\Delta$, as $A \to B \in \Theta$ and $A \in \Sigma$ gives $(B \to B) \to (A \to B) \in \Theta$, by $\mathbf{B}'$. As $B \to B \in \Gamma$ and $R\Gamma\Theta\Xi$, we have $A \to B \in \Xi$. This, along with $R\Sigma\Xi\Delta$ and $A \in \Sigma$ gives $B \in \Delta = \Omega'$, as we desired.

$\mathbf{B}$. Assume that $\mathbf{B}$ holds, and that Σ, Θ, Γ, Δ and Ξ are prime Π-theories such that $R\Sigma\Xi\Delta$ and $R\Gamma\Theta\Xi$. Set Ω to be $\{B : (\exists A)(A \to B \in \Gamma) \wedge (A \in \Sigma)\}$, and then it is clear that $R\Sigma\Gamma\Omega$. (Even in the case where $\Gamma = \Pi$, in which case $\Omega = \Theta$, and we have our result.) To show that $R\Omega\Theta\Delta$, consider $A \to B \in \Theta$ and $A \in \Omega$; there must be a $C \in \Sigma$ where $C \to A \in \Gamma$. So as $\vdash_\Pi (A \to B) \to ((C \to A) \to (C \to B))$, we have $(C \to A) \to (C \to B) \in \Theta$. $R\Gamma\Theta\Xi$ then gives $(C \to B) \in \Xi$, and $R\Sigma\Xi\Delta$ with $C \in \Sigma$ gives $B \in \Delta$, as we wished. Ω is a Π-theory, and can be primed by a priming lemma, in the usual manner.

$\mathbf{W}$. Assume that $\mathbf{W}$ holds, and let Σ, Γ and Δ be arbitrary Π-theories where $R\Sigma\Gamma\Delta$. Let $\Omega = \{B : (\exists A)(A \to B \in \Gamma) \wedge (A \in \Sigma)\}$, so $R\Sigma\Gamma\Omega$ is immediate. (Except if $\Gamma = \Pi$, in which case $\Delta = \Sigma$, and we simply set $\Omega = \Sigma$ as well. It follows that $R\Sigma\Omega\Delta$, as $R\Sigma\Sigma\Sigma$, because $\mathbf{WI}$ is a consequence of $\mathbf{W}$. This gives the result in this case.) To show that $R\Sigma\Omega\Delta$ let $A \to B \in \Omega$ and $A \in \Sigma$, so there is a $C \in \Sigma$ where $C \to (A \to B) \in \Gamma$. Thus, $A \wedge C \in \Sigma$, and $A \wedge C \to B \in \Gamma$, as the following derivation shows.

$\vdash_\Pi A \wedge C \to A, \quad A \wedge C \to C,$	
$\vdash_\Pi (A \to B) \to (A \wedge C \to B),$	by suffixing,
$\vdash_\Pi (C \to (A \to B)) \to (C \to (A \wedge C \to B)),$	by prefixing,
$\vdash_\Pi (C \to (A \wedge C \to B)) \to (A \wedge C \to (A \wedge C \to B)),$	by suffixing,
$\vdash_\Pi (C \to (A \to B)) \to (A \wedge C \to (A \wedge C \to B)),$	by transitivity.

This ensures that $A \wedge C \to (A \wedge C \to B) \in \Gamma$, and $\mathbf{W}$ gives $A \wedge C \to B \in \Gamma$. So $R\Sigma\Gamma\Delta$ gives $B \in \Delta$ as desired. The usual application of the priming lemma gives a prime Π-theory Ω' with the desired properties, except if $\Omega' = \Pi$ as sets, and it is taken to be Π. In that case, $R\Gamma\Omega'\Delta$ is not assured, for it is not clear that $\Gamma = \Delta$. However, if this is the case, we can take Ω' to be Π' and all is well.

$\mathbf{CI}$. Assume that $\mathbf{CI}$ holds. We wish to show that in an almost canonical model structure, $R\Sigma\Gamma\Delta \Rightarrow R\Gamma\Sigma\Delta$. However, in the case of $\Sigma = \Pi$ this will not hold in general. We have $R\Pi\Gamma\Delta$ whenever $\Gamma \subseteq \Delta$ (as is easy to check, using $\mathbf{CI}$) but $R\Gamma\Pi\Delta$ only when $\Gamma = \Delta$.

We will work in the almost-canonical model of Π-theories as usual, but with a different relation R' defined to meet this objection: $R'\Sigma\Gamma\Delta$ if and only if for each $A \to B \in \Gamma$, if $A \in \Sigma$ then $B \in \Delta$ for $\Sigma, \Gamma \neq \Pi$. Otherwise, $R'\Pi\Gamma\Delta$ and $R'\Gamma\Pi\Delta$ if and only if $\Gamma = \Delta$. We need to show that in this model of Π-theories that $\Sigma \models A$ if and only if $A \in \Sigma$, and that it satisfies $R'\Sigma\Gamma\Delta \Rightarrow R'\Gamma\Sigma\Delta$. The latter part is simpler.

If either of Σ or Γ is Π, then the condition is satisfied by fiat. If Σ and Γ are both not Π, then let $A \to B \in \Sigma$, $A \in \Gamma$ and $R'\Sigma\Gamma\Delta$. **CI** gives $(A \to B) \to B \in \Gamma$, and $R'\Sigma\Gamma\Delta$ then gives $B \in \Delta$, and so we have $R'\Gamma\Sigma\Delta$, as desired.

To show that the model structure satisfies $\Gamma \models A$ if and only if $A \in \Gamma$, we need only consider the case where A is $C \to D$, and where Γ is not Π. The rest of the proof is unaltered from the almost-canonical structure. We need to show that $\Gamma \models C \to D$ if and only if $C \to D \in \Gamma$. From right to left, it is enough to note that $C \to D \in \Gamma$ ensures that for all prime Π-theories Σ and Δ where $R\Sigma\Gamma\Delta$, if $C \in \Sigma$, then $D \in \Delta$, by the definition of R. As $R\Sigma\Gamma\Delta \Rightarrow R'\Sigma\Gamma\Delta$, we have that all prime Π-theories Σ and Δ where $R'\Sigma\Gamma\Delta$, if $C \in \Sigma$, then $D \in \Delta$.

And in the other direction, if $\Gamma \models C \to D$, then for each Σ, Δ where $R'\Sigma\Gamma\Delta$, if $C \in \Sigma$ then $D \in \Delta$. If $C \to D \notin \Gamma$, then by Lemma 5.12 there are prime Π-theories Σ and Δ where $R\Sigma\Gamma\Delta$, $C \in \Sigma$ and $D \notin \Delta$. In this case, $R'\Sigma\Gamma\Delta$ unless $\Sigma = \Pi$ (the case where we've been mucking about with R'). In this case, if $R\Pi\Gamma\Delta$, and $A \in \Gamma$, then $\vdash_\Pi A \to ((A \to A) \to A)$ gives $(A \to A) \to A \in \Gamma$, which with $R\Pi\Gamma\Delta$ and $A \to A \in \Pi$ gives $A \in \Delta$. So $\Gamma \subseteq \Delta$, which means that $D \notin \Gamma$, and as $R'\Pi\Gamma\Gamma$, we have our result, that not all Π-theories Σ and Δ where $R'\Sigma\Gamma\Delta$ satisfy $C \in \Sigma \Rightarrow D \in \Delta$. Contraposing gives us the desired result.

C. **C** is a stronger version of **CI**, so we need R' in this case too. Assume that **C** holds, and let Σ, Γ, Δ, Θ and Ξ be Π-theories such that $R'\Sigma\Delta\Xi$ and $R'\Gamma\Theta\Delta$. Define $\Omega = \{B : (\exists A)(A \to B \in \Theta) \wedge (A \in \Sigma)\}$; this satisfies $R\Sigma\Theta\Omega$ by definition.

If $\Theta = \Pi$, then $\Sigma = \Omega$ and all is well. If $\Sigma = \Pi$, then $\Omega = \Theta$, because if $B \in \Theta$, then $(B \to B) \to B \in \Theta$ too, by **C** (Derive **CI** from **C**, and this is enough.), so as $B \to B \in \Pi$, $\Theta \subseteq \Omega$. Conversely, if $B \in \Omega$, then $B \in \Theta$, as Θ is a Π-theory. So, we have $R'\Sigma\Theta\Omega$ in any case.

To show that $R'\Gamma\Omega\Xi$, let $A \to B \in \Omega$ and $A \in \Gamma$. There is a $C \in \Sigma$ where $C \to (A \to B) \in \Theta$. This gives $A \to (C \to B) \in \Theta$ by **C**, and hence $C \to B \in \Delta$ as $R\Gamma\Theta\Delta$. This, with $R\Sigma\Delta\Xi$ and $C \in \Sigma$ gives $B \in \Xi$, and hence $R\Gamma\Omega\Xi$, as desired. We want $R'\Gamma\Omega\Xi$. If $\Omega = \Pi$ as sets, then take $\Omega = \Pi'$, and so $R'\Gamma\Omega\Xi$. If $\Gamma = \Pi$, then $\Theta = \Delta$, and as $R'\Sigma\Delta\Xi$, we have $R'\Sigma\Theta\Xi$, and we are safe to take Ξ for Ω. In this case $R'\Sigma\Theta\Xi$, as $\Xi \supseteq \Omega$, and $R'\Gamma\Xi\Xi$ A priming lemma gives a prime Π-theory Ω', with the desired properties.

S. We're back to R. Let Σ, Θ, Γ, Δ and Ξ be arbitrary prime Π-theories such that $R\Sigma\Delta\Xi$ and $R\Gamma\Theta\Delta$. Let $\Psi = \{B : (\exists A)(A \to B \in \Theta) \wedge (A \in \Sigma)\}$ and $\Phi = \{B : (\exists A)(A \to B \in \Gamma) \wedge (A \in \Sigma)\}$. Then it is immediate that $R\Sigma\Theta\Psi$ and $R\Sigma\Gamma\Phi$. (Even when $\Sigma = \Pi$, for in that case $\Psi = \Theta$, and if $\Gamma = \Pi$, $\Phi = \Sigma$).

It remains for us to see that $R\Phi\Psi\Xi$. To see this, let $A \to B \in \Psi$ (so there is a $C \in \Sigma$ where $C \to (A \to B) \in \Theta$) and $A \in \Phi$ (so there is a $D \in \Sigma$ where $D \to A \in \Gamma$). We then see that $C \wedge D \in \Sigma$, $C \wedge D \to (A \to B) \in \Theta$ (by prefixing), and $C \wedge D \to A \in \Gamma$ (also by prefixing). But **S** gives $(D \wedge C \to A) \to (D \wedge C \to B) \in \Theta$, and so $R\Gamma\Theta\Delta$ ensures that $D \wedge C \to B \in \Delta$. This, in turn gives $B \in \Xi$, as $R\Sigma\Delta\Xi$. The result follows from an application Lemma 5.11 to Ψ and Φ. If $\Phi' = \Pi$ as sets, then take Φ' to be Π', and the result that $R\Phi'\Psi'\Xi$ is then preserved.

CS. Assume **CS**, and let Σ, Γ, Δ, Θ and Ξ be arbitrary Π-theories such that $R\Sigma\Delta\Xi$ and $R\Gamma\Theta\Delta$. Let $\Phi = \{B : (\exists A)(A \to B \in \Theta) \wedge (A \in \Sigma)\}$ and $\Psi = \{B : (\exists A)(A \to B \in \Gamma) \wedge (A \in \Sigma)\}$, so $R\Sigma\Gamma\Psi$ and $R\Sigma\Theta\Phi$ are immediate. (If $\Gamma = \Pi$, $\Psi = \Sigma$, and if $\Theta = \Pi$, $\Phi = \Sigma$.)

We have only to demonstrate that $R\Phi\Psi\Theta$ (as priming lemmas give us the rest of the result). To show this, let $A \to B \in \Psi$ (so there is a $C \in \Sigma$ where $C \to (A \to B) \in \Gamma$) and $A \in \Phi$ (so there is a $D \in \Sigma$ where $D \to A \in \Theta$). This ensures that $C \wedge D \in \Sigma$, and that, by prefixing, $C \wedge D \to (A \to B) \in \Gamma$ and $C \wedge D \to A \in \Theta$. **CS** then gives $(C \wedge D \to (A \to B)) \to (C \wedge D \to B) \in \Theta$, and $R\Gamma\Theta\Delta$ gives $C \wedge D \to B \in \Delta$, and $R\Sigma\Delta\Xi$ gives $B \in \Xi$, as we set out to show. Lemma 5.11 completes the proof. Again, if $\Phi' = \Pi$ as sets, then set $\Phi' = \Pi'$, and the result that $R\Phi'\Psi'\Theta$ is preserved.

$\mathbf{C}''$. Assume that $\mathbf{C}''$ holds, and let Σ be a prime Π-theory. We wish to show that $R\Pi\Sigma\Sigma$, so let $A \to B \in \Sigma$ and $A \in \Pi$. By $\mathbf{C}''$, $(A \to B) \to B \in \Pi$, and $R\Sigma\Pi\Sigma$ gives $B \in \Sigma$ as we desired.

This completes the list, and our equivalences have been shown. ◁

5.6 An Ordering on Indices

Routley gives more axioms to extend $\mathbf{DW}^+$ along with their corresponding restriction on the relation R [132, 133]. As an example, $B \to (A \to B)$ is shown to correspond to the condition $Rabc \Rightarrow b \sqsubseteq c$. The relation $\sqsubseteq$ on indices needs some explanation, as we have not introduced it yet. Simply put, in the unsimplified semantics $a \sqsubseteq b$ if and only if $Ragb$, where g is the base index. (Or in the case of more than one base index, $a \sqsubseteq b \Longleftrightarrow Raxb$ for *some* base index x.) In that semantics, this has the pleasing property of ensuring that if A is true in a, then A is true in b. Its corresponding condition in the canonical model structure is represented by the relation of containment, that is, $\Sigma \sqsubseteq \Delta \Longleftrightarrow \Sigma \subseteq \Delta$. Unfortunately, in the simplified semantics such a connection does not exist, for we have $Ragb$ if and only if $a = b$, so the definition of $\sqsubseteq$ collapses into identity. You may think that the occurrences of $\sqsubseteq$ in modelling conditions could be replaced by $=$, but this fails in general. For example, the class of simplified interpretations satisfying $Rabc \Rightarrow b = c$ is certainly sound with respect to the axiom $B \to (A \to B)$, but completeness fails. We would need to show that in the almost-canonical model, $R\Sigma\Gamma\Delta \Rightarrow \Gamma = \Delta$, which, when $\Gamma = \Pi$, ensures that $\Pi = \Delta$ for each Π-theory Δ, and thus there is only one index. The condition on R is too strict, and we need to find another way to model the relation $\sqsubseteq$.

The way to proceed seems to be as follows. We can define $\sqsubseteq$ as a *primitive* binary relation on indices, with conditions that are relatively simple to check practically. Then we can show that this relation has the desired properties (namely that $a \sqsubseteq b \Rightarrow (a \models A \Rightarrow b \models A)$ for each formula A), and define an *extended interpretation* to be an interpretation with such an additional binary relation. Then the extra modelling results hold for extended interpretations. This is what we shall do.

Definition 5.7 Given an interpretation $\langle g, W, R, \models\rangle$, if binary relation $\sqsubseteq$ on W satisfies

$$a \sqsubseteq b \Rightarrow \begin{cases} a \models p \Rightarrow b \models p & \text{for every propositional variable } p, \\ Rcbd \Rightarrow Rcad & \text{if } a \neq g, \\ Rcbd \Rightarrow c \sqsubseteq d & \text{if } a = g. \end{cases}$$

it is said to be a *containment* relation on $\langle g, W, R, \models\rangle$.

We can then prove the following result:

THEOREM 5.15 *Given a containment relation* $\sqsubseteq$ *on* $\langle g, W, R, \models\rangle$, $a \sqsubseteq b \Rightarrow (a \models A \Rightarrow b \models A)$ *for every formula* A.

Proof: We will prove this by induction on the complexity of formulae. The result holds (for all indices a and b where $a \sqsubseteq b$) for propositional variables, and the inductive cases for $\wedge$ and $\vee$ are immediate. Now assume that the result holds for A and B, that $a \sqsubseteq b$, and $a \models A \to B$.

If $a \neq g$ then we have that for all c and d where $Rcad$, $c \models A \Rightarrow d \models B$, and as $Rcbd \Rightarrow Rcad$, we have that for all c and d where $Rcbd$, $c \models A \Rightarrow d \models B$, and hence $b \models A \to B$.

If $a = g$, then for each c, $c \models A \Rightarrow c \models B$. We wish to show that $b \models A \to B$. We have by the condition on $\sqsubseteq$ that $Rcbd \Rightarrow c \sqsubseteq d$, so for each c and d where $Rcbd$, if $c \models A$ then $c \models B$ (as $g \models A \to B$), which gives $d \models B$ (as $Rcbd$ gives $c \sqsubseteq d$). Hence $b \models A \to B$. This completes the proof. $\triangleleft$

Now that we have $\sqsubseteq$, we are able to model t. It has the obvious condition

- $x \models t \Longleftrightarrow g \sqsubseteq x$.

It it simple to show that this soundly models the t rules. If $g \models A$ then for all $x \geq g$, $x \models A$ too. So, for each x where $x \models t$, $x \models A$; which means that $g \models t \to A$ as desired. Conversely, if $g \models t \to A$, $g \models t$ gives $g \models A$. We leave completeness for this rule until the completeness proofs for other logics.

We can also use this relation to prove soundness of further extensions of $\mathbf{DW}^+$. These are catalogued in the following theorem.

THEOREM 5.16 *For each row in the list below, the logic* $\mathbf{DW}^+$ *with an axiom added is sound with respect to the class of extended* $\mathbf{DW}^+$ *interpretations* $\langle g, W, R, \models, \sqsubseteq\rangle$ *where*

R *satisfies the corresponding condition.*

K	$B \to (A \to B)$	$Rabc \Rightarrow b \sqsubseteq c$
K′	$A \to (B \to B)$	$Rabc \Rightarrow a \sqsubseteq c$
K²	$A \to (B \to (C \to A))$	$R^2a(bc)d \Rightarrow c \sqsubseteq d$
Exp	$(A \wedge B \to C) \to (A \to (B \to C))$	$R^2a(bc)d \Rightarrow$ for some x $a \sqsubseteq x$, $b \sqsubseteq x$ & $Rxcd$
Ord	$(A \to B) \vee (B \to A)$	$a \sqsubseteq b$ or $b \sqsubseteq a$
Mingle	$A \to (A \to A)$	$Rabc \Rightarrow a \sqsubseteq c$ or $b \sqsubseteq c$
StOrd	$(A \wedge B \to C) \to ((A \to C) \vee (B \to C))$	$Rabc$ & $Ra'bc' \Rightarrow$ for some x $a \sqsubseteq x$, $a' \sqsubseteq x$ & $(Rxbc$ or $Rxbc')$

Table 5.2

Proof: We proceed exactly as in the previous collection of soundness results, except for using the fact that $a \sqsubseteq b$ gives $a \models A \Rightarrow b \models A$ for any formula A. The proofs are in tableau form.

K. Assume that R satisfies $Rabc \Rightarrow b \sqsubseteq c$.

1	$g \not\models B \to (A \to B)$	
2	$x \models B$	[1]
3	$x \not\models A \to B$	[1]
4	$Ryxz$	[3]
5	$y \models A$	[3]
6	$z \not\models B$	[3]
7	$x \sqsubseteq z$	[4,K]
8	$z \models B$	[2,7]
	$\times$	[6,8]

K′. Assume that R satisfies $Rabc \Rightarrow a \sqsubseteq c$ for each $a, b, c \in W$. Then,

1	$g \models A \to (B \to B)$	
2	$x \to A$	[1]
3	$x \not\models B \to B$	[1]
4	$Ryxz$	[3]
5	$y \models B$	[3]
6	$z \not\models B$	[3]
7	$y \sqsubseteq z$	[4,K']
8	$z \models B$	[6,7]
	$\times$	[6,8]

$\mathbf{K}^2$. Assume that $R^2a(bc)d \Rightarrow c \sqsubseteq d$.

$$
\begin{array}{llr}
1 & g \not\models A \to (B \to (C \to A)) & \\
2 & w \models A & [1] \\
3 & w \not\models B \to (C \to A) & [1] \\
4 & Rxwy & [3] \\
5 & x \models B & [3] \\
6 & y \not\models C \to A & [3] \\
7 & Rzyu & [6] \\
8 & z \models C & [6] \\
9 & u \not\models A & [6] \\
10 & w \sqsubseteq u & [4,7,\mathrm{K}^2] \\
11 & u \models A & [2,10] \\
 & \times & [9,11]
\end{array}
$$

Exp. Assume that $R^2a(bc)d \Rightarrow$ for some x, $a \sqsubseteq x$, $b \sqsubseteq x$ and $Rxcd$.

$$
\begin{array}{llr}
1 & g \not\models (A \wedge B \to C) \to (A \to (B \to C)) & \\
2 & x \models A \wedge B \to C & [1] \\
3 & x \not\models A \to (B \to C) & [1] \\
4 & Ryxz & [3] \\
5 & y \models A & [3] \\
6 & z \not\models B \to C & [3] \\
7 & Ruzv & [6] \\
8 & u \models B & [6] \\
9 & v \not\models C & [6] \\
10 & u \sqsubseteq w,\ y \sqsubseteq w,\ Rwxv & [4,7,\mathrm{Exp}] \\
11 & w \models A \wedge B & [5,8,9] \\
12 & v \models C & [2,10,11] \\
 & \times & [9,12]
\end{array}
$$

Ord. Assume that for each a and b either $a \sqsubseteq b$ or $b \sqsubseteq a$. Then

$$
\begin{array}{llr}
1 & g \not\models (A \to B) \vee (B \to A) & \\
2 & g \not\models A \to B & [1] \\
3 & g \not\models B \to A & [1] \\
4 & x \models A & [2] \\
5 & x \not\models B & [2] \\
6 & y \models B & [3] \\
7 & y \not\models A & [3]
\end{array}
$$

$$
\begin{array}{llr|llr}
8 & x \sqsubseteq y & & 9 & y \sqsubseteq x & [\mathrm{Ord}] \\
10 & y \models A & [4,8] & 11 & x \models B & [6,9] \\
 & \times & [4,10] & & \times & [6,11]
\end{array}
$$

Mingle. Assume that $Rabc \Rightarrow a \sqsubseteq c$ or $b \sqsubseteq c$.

$$\begin{array}{lcl}
1 & g \not\models A \to (A \to A) & \\
2 & x \models A & [1] \\
3 & x \not\models A \to A & [1] \\
4 & Ryxz & [3] \\
5 & y \models A & [3] \\
6 & z \not\models A & [3]
\end{array}$$

$$\begin{array}{llllll}
7 & x \sqsubseteq z & & 8 & y \sqsubseteq z & [4,\text{Mingle}] \\
9 & z \models A & [2,7] & 10 & z \models A & [5,8] \\
 & \times & [6,9] & & \times & [6,10]
\end{array}$$

StOrd. Assume that $Rabc$ and $Ra'bc' \Rightarrow$ for some x, $a \sqsubseteq x$, $a' \sqsubseteq x$ and $Rxbc$ or $Rxbc'$.

$$\begin{array}{lccl}
1 & \multicolumn{2}{c}{g \not\models (A \wedge B \to C) \to (A \to C) \vee (B \to C)} & \\
2 & \multicolumn{2}{c}{x \models A \wedge B \to C} & [1] \\
3 & \multicolumn{2}{c}{x \not\models (A \to C) \vee (B \to C)} & [1] \\
4 & \multicolumn{2}{c}{x \not\models A \to C} & [3] \\
5 & \multicolumn{2}{c}{x \not\models B \to C} & [3] \\
6 & Ryxz & Ry'xz' & [4,5] \\
7 & y \models A & y' \models B & [4,5] \\
8 & z \not\models C & z' \not\models C & [4,5] \\
9 & y \sqsubseteq u & y' \sqsubseteq u & [6,\text{StOrd}]
\end{array}$$

$$\begin{array}{llllll}
10 & Ruxz & & 11 & Ruxz' & [6,\text{StOrd}] \\
12 & u \models A \wedge B & [7,9] & 13 & u \models A \wedge B & [7,9] \\
14 & z \models C & [2,10,12] & 15 & z' \models C & [2,10,13] \\
 & \times & [8,14] & & \times & [8,15]
\end{array}$$

◁

For the completeness proof we need a containment relation in the canonical models. Thankfully the obvious candidate works.

THEOREM 5.17 *In the canonical model, and in the almost canonical model,* $\subseteq$ *is a containment relation. In this model, the* t*-condition is satisfied if the logic includes* t.

Proof: That $\Sigma \subseteq \Gamma \Rightarrow (p \in \Sigma \Rightarrow p \in \Gamma)$ is immediate. If $\Sigma \subseteq \Gamma$ and $\Sigma \neq \Pi$ then $R\Delta\Gamma\Phi \Rightarrow R\Delta\Sigma\Phi$ by the definition of R, and if $R\Delta\Gamma\Phi$, and $\Pi \subseteq \Gamma$, then each formula $A \to A \in \Gamma$, and hence $\Delta \subseteq \Phi$.

Clearly $\vdash_\Pi t$, so $t \in \Pi$, and moreover, for any $\Gamma \supseteq \Pi$, $t \in \Gamma$. Conversely, if $t \in \Gamma$, take $A \in \Pi$. Then $\vdash_\Pi t \to A$ ensures that $A \in \Gamma$, and hence, $\Pi \subseteq \Gamma$ as desired. ◁

One further result we need is that for certain extensions of $\mathbf{DW}^+$, we can do without the empty and full Π-theories, and still have an interpretation. (The collection of *all* formulae is the full Π-theory.) These two rather excessive theories are appropriately called degenerate theories, and this result is called a *non-degeneracy theorem*.

THEOREM 5.18 *Provided that* $A \to A \in \Sigma$ *for each formula* A *and each non-empty prime* Π*-theory* Σ*, then the canonical (or almost-canonical) interpretation, which is limited to non-degenerate prime* Π*-theories is an interpretation of* $\mathbf{DW}^+$.

Proof: To show that this structure *is* an interpretation, it is sufficient to show that the assignment $\Sigma \models A$ iff $A \in \Sigma$ satisfies the inductive characterisation of an interpretation. Because the structure is a *reduction* of the earlier structure, inductive cases are exactly the same, except for showing that when $A \to B \notin \Sigma$ (for non-degenerate Σ), there are non-degenerate prime Γ and Δ where $R\Gamma\Sigma\Delta$, $A \in \Gamma$ and $B \notin \Delta$. To this end, define $\Gamma' = \{C : \vdash_\Pi A \to C\}$ and $\Delta' = \{D : (\exists C)(C \in \Gamma' \text{ and } C \to D \in \Sigma)\}$. We will show that $A \in \Gamma' \cap \Delta'$ and $B \notin \Gamma' \cup \Delta'$, so that these theories are non-degenerate.

First note that $\nvdash_\Pi A \to B$. For otherwise we have $\vdash_\Pi (A \to A) \to (A \to B)$ by prefixing, and $A \to A \in \Sigma$ gives $A \to B \in \Sigma$, which we know does not obtain. So it follows that $B \notin \Gamma'$. That $A \to A \in \Sigma$ and $A \in \Gamma'$ gives $A \in \Delta'$, as we desired. Noting that $B \notin \Delta'$ completes the first part of the result — Γ' and Δ' are non-degenerate.

We only need to find non-degenerate prime Γ and Δ to complete the theorem. This is done by applying Lemma 5.9 — we need just show that the Γ and Δ so obtained are non-degenerate. As $\Gamma' \subseteq \Gamma$, Γ is non-empty. To see that $A \notin \Gamma$, note that in the proof Γ is disjoint with Θ_2, and as $A \to A \in \Sigma$, $A \in \Theta_2$, giving $A \notin \Gamma$. The result of the lemma ensures that $B \notin \Delta$ and that $\Delta' \subseteq \Delta$, so Δ is also non-degenerate.

If $\Sigma = \Pi$, then $\Delta' = \Gamma'$, and noting that $R\Delta\Sigma\Delta$ (where Δ was constructed by Lemma 5.9), $A \in \Delta$ and $B \notin \Delta$ is sufficient to complete the proof. $\triangleleft$

We now give some example conditions which enable us to use non-degenerate models.

THEOREM 5.19 *Conditions* $\mathbf{K}$, $\mathbf{K}'$ *and* $\mathbf{K}^2$ *ensure that* $A \to A \in \Sigma$ *for each non-empty prime* Π*-theory* Σ.

Proof: $\mathbf{K}'$ is obvious. For $\mathbf{K}$, note that $\vdash_\Pi (A \to A) \to (B \to (A \to A))$ is an instance of $\mathbf{K}$, and hence $\vdash_\Pi B \to (A \to A)$. For $\mathbf{K}^2$, note that $\vdash_\Pi (A \to A) \to ((A \to A) \to (B \to (A \to A)))$ is an instance of $\mathbf{K}^2$, and hence $\vdash_\Pi B \to (A \to A)$. $\triangleleft$

This gives us enough machinery to prove completeness for the rest of the positive extensions of $\mathbf{DW}^+$. They are of the same form as the other completeness proofs, except that they use the fact that $\subseteq$ is a containment relation in the canonical model and in the almost canonical model.

THEOREM 5.20 *For each row in Table 5.2 the logic* $\mathbf{DW}^+$ *with an axiom added is complete with respect to the class of extended* $\mathbf{DW}^+$ *interpretations* $\langle g, W, R, \models, \sqsubseteq\rangle$ *where* R *satisfies the corresponding condition.*

Proof: We take these individually as before, using the almost canonical model:

$\mathbf{K}$. Assume that $\mathbf{K}$ holds. We can use the non-degenerate model, by Theorem 5.19. Assume that $R\Sigma\Gamma\Delta$. Take some $A \in \Sigma$ and some $B \in \Gamma$, then $\mathbf{K}$ gives $A \to B \in \Gamma$ and hence $B \in \Delta$. This means that $\Gamma \subseteq \Delta$ as desired. The result holds, even if $\Sigma = \Pi$.

$\mathbf{K}'$. Assume that $\mathbf{K}'$ holds. We can use the non-degenerate model, by Theorem 5.19. Assume also that Σ, Γ, Δ are non-degenerate Π-theories satisfying $R\Sigma\Gamma\Delta$. We wish to show that $\Sigma \subseteq \Delta$. This is immediate for the case $\Gamma = \Pi$, and otherwise, note that for some A, $A \in \Gamma$, and hence $B \to B \in \Gamma$ for each B, so $B \in \Sigma$ gives $B \in \Delta$, as we desired.

$\mathbf{K}^2$. Assume $\mathbf{K}^2$. We can use the non-degenerate model, by Theorem 5.19. Then take $R\Sigma\Gamma\Delta$ and $R\Theta\Delta\Xi$. Take $A \in \Gamma$, $B \in \Sigma$ and $C \in \Theta$. $\mathbf{K}^2$ ensures that $B \to (C \to A) \in \Gamma$, and hence $A \in \Xi$, as desired (even if $\Gamma = \Pi$).

Exp. Assume **Exp**, and that $R\Sigma\Gamma\Delta$ and $R\Theta\Delta\Xi$. We wish to find a prime Π-theory Φ' where $R\Phi'\Gamma\Xi$ and both $\Sigma, \Theta \subseteq \Phi'$. To this end, set $\Phi = \{A : (\exists C \in \Sigma, D \in \Theta) \vdash_\Pi C \wedge D \to A\} \cup \Sigma \cup \Theta$. It is clear that $\Sigma, \Theta \subseteq \Phi$, and to show that Φ is a Π-theory, note that if Σ and Θ are both non-empty, then $\Phi = \{A : (\exists C \in \Sigma, D \in \Theta) \vdash_\Pi C \wedge D \to A\}$, and this is clearly a Π-theory, as Σ, Θ are both Π-theories, and $\to$ is transitive (in that if $\vdash_\Pi A \to B$ and $\vdash_\Pi B \to C$ then $\vdash_\Pi A \to C$). So if both Θ and Σ are non-empty, Φ is a Π-theory. Otherwise (if one of Θ and Σ are empty), Φ is the union of Θ and Σ, which is then also a Π-theory.

To show that $R\Phi\Gamma\Xi$, let $A \to B \in \Gamma$ and $A \in \Phi$. By definition, there are $C \in \Sigma$ and $D \in \Theta$ where $\vdash_\Pi C \wedge D \to A$. Hence, $\vdash_\Pi (A \to B) \to (C \wedge D \to B)$ and so $C \wedge D \to B \in \Gamma$, which by **Exp** gives $C \to (D \to B) \in \Gamma$. $R\Sigma\Gamma\Delta$ and $R\Theta\Delta\Xi$ then give us $B \in \Xi$ as desired, so $R\Phi\Gamma\Xi$, if $\Gamma \neq \Pi$.

A priming lemma then completes the proof except for the case where $\Sigma = \Pi$. In that case, instead of using Φ, we simply need to show that $\Sigma, \Theta \subseteq \Xi$. Note that $\Gamma = \Pi$ gives $\Sigma = \Delta$, and hence we have $R\Theta\Sigma\Xi$ For this it is sufficient to note that as $\vdash_\Pi (A \wedge B \to A) \to (A \to (B \to A))$ and $\vdash_\Pi (A \wedge B \to B) \to (A \to (B \to B))$, **Exp** gives both $\mathbf{K}$ and $\mathbf{K}'$, which in turn ensures that $R\Theta\Sigma\Xi$ gives $\Sigma, \Theta \subseteq \Xi$ by our results from before.

Ord. Assume **Ord** and take Π-theories Σ and Γ where $\Sigma \not\subseteq \Gamma$. Hence there is some B where $B \in \Sigma$ and $B \notin \Gamma$. Given $A \in \Gamma$, it is sufficient to show that $A \in \Sigma$. As $A \to B \notin \Pi$ (since $A \in \Gamma$ and $B \notin \Gamma$), and $(A \to B) \vee (B \to A) \in \Pi$ we must have $B \to A \in \Pi$, which with $B \in \Sigma$ gives us our result.

Mingle. Assume **Mingle**, and that $R\Sigma\Gamma\Delta$. We wish to show that either $\Sigma \subseteq \Delta$ or $\Gamma \subseteq \Delta$. To show that this obtains, take $A \in \Sigma$ where $A \notin \Delta$ and $B \in \Gamma$ where $B \notin \Delta$. Then $A \vee B \notin \Delta$, but $A \vee B \in \Sigma, \Gamma$. **Mingle** gives $A \vee B \to A \vee B \in \Gamma$, and $R\Sigma\Gamma\Delta$ gives us $A \vee B \in \Delta$. Hence our result.

StOrd. Assume **StOrd**, and that $R\Sigma\Gamma\Delta$ and $R\Sigma'\Gamma\Delta'$. We wish to find a prime Π-theory Φ' where $\Sigma, \Sigma' \subseteq \Phi'$, and either $R\Phi'\Gamma\Delta$ or $R\Phi'\Gamma\Delta'$. Routley recommends [134] that to this end we define a set $\Phi = \{A : (\exists C \in \Sigma, C' \in \Sigma') \vdash_\Pi C \wedge C' \to A\}$ and show that either $R\Phi\Gamma\Delta$ or $R\Phi\Gamma\Delta'$. To do this you take $A \to B \in \Gamma$ and $A \in \Phi$. Then $\vdash_\Pi C \wedge C' \to A$ for some C, C' in Σ, Σ' respectively. So as $\vdash_\Pi (A \to B) \to (C \wedge C' \to B)$ we see that $C \wedge C' \to B \in \Gamma$, and hence $(C \to B) \vee (C' \to B) \in \Gamma$, giving either $C \to B$ or $C' \to B$ in Γ. $R\Sigma\Gamma\Delta$ and $R\Sigma'\Gamma\Delta'$ then gives either $B \in \Delta$ or $B \in \Delta'$. And the text leaves us there. The astute will note that this is *not* enough to give us the result, as for a range of values B, there is nothing to ensure that they land in the same place. Some might end up in

Δ, and some in Δ'. All we have shown is that $R\Sigma\Theta'(\Delta \cup \Delta')$. Fortunately, all is not lost, as **StOrd** gives **Ord**, as $\vdash_\Pi (A \wedge B \to A \wedge B) \to (A \to A \wedge B) \vee (B \to A \wedge B)$, so $\vdash_\Pi (A \to A \wedge B) \vee (B \to A \wedge B)$ which easily yields $\vdash_\Pi (A \to B) \vee (B \to A)$, as we wished. So, by the proof for **Ord**, it follows that either $\Delta' \subseteq \Delta$, or $\Delta \subseteq \Delta'$, so $\Delta \cup \Delta'$ is one of them, giving the result.

For those who prefer a smoother proof, abandon all thoughts of Φ, and take the larger of Γ and Γ' as our required prime Π-theory. The result follows immediately. $\triangleleft$

Routley suggests a few more extensions — such as $A \vee (A \to B)$ — these to seem require the non-degenerate model structure to push through the completeness proofs, but it seems that Theorem 5.19 cannot be proved for these extensions, despite what is said in *Relevant Logics and their Rivals* [131]. There, on page 314, non-degeneracy is assumed for this axiom, but on page 317, it is only shown to work for axioms like our **K**.[4]

Despite this setback, it is possible to extend the structure of an interpretation yet again, by adding an explicit empty index e, satisfying certain obvious conditions. Then a phrase like $a \neq e$ is used in a modelling condition whenever it is needed that a be non-empty. The details of this approach can be found on page 380 [131], and the interested reader is referred there. We will extend the semantics to deal with a more pressing need, and that is to add negation.

5.7 Adding Negation

The addition of negation to the story complicates things somewhat. Priest and Sylvan show that there are (at least) two different ways of expanding the simplified semantics to deal with negation [118]. In this section we will show that the semantics using the Routley '$*$' operation can model common negation extensions of **B**, the 'basic' logic modelled in Priest and Sylvan's work. **BM** is a weak logic extending $\mathbf{DW}^+$ is obtained by adding the rule:

$$A \to B \vdash {\sim} B \to {\sim} A \qquad \textbf{CPR}$$

along with the de Morgan laws

$${\sim}(A \vee B) \leftrightarrow {\sim} A \wedge {\sim} B \qquad \textbf{DM1}$$

$${\sim}(A \wedge B) \leftrightarrow {\sim} A \vee {\sim} B \qquad \textbf{DM2}$$

to $\mathbf{DW}^+$. Priest and Sylvan show that if we extend interpretations to contain a function $*\colon W \to W$, and define the truth conditions for negation as

$$w \models {\sim} A \iff w^* \not\models A$$

the logic **BM** is sound with respect to these conditions. To show completeness, define $*$ on the set of prime Π-theories by setting $\Sigma^* = \{A : {\sim} A \not\in \Sigma\}$.

This is shown to send prime Π-theories to prime Π-theories, and to give the desired results. The details of the completeness proof are not difficult, and the interested reader is referred to Priest and Sylvan for the details.

The system **B** can be obtained from **BM** by adding the axiom:

$$A \leftrightarrow {\sim}{\sim} A \qquad \textbf{DN}$$

(Or alternatively, add to $\mathbf{DW}^+$ ${\sim}{\sim} A \to A$ and the rule $A \to {\sim} B \vdash B \to {\sim} A$.) To obtain semantics for **B** we simply require that $*$ satisfy $w^{**} = w$ in each interpretation. Soundness and completeness are simple to show. The only other construction we need to consider is the containment relation $\sqsubseteq$ on indices. It no longer follows that containment relations as they stand satisfy the condition $a \sqsubseteq b \Rightarrow (a \models A \Rightarrow b \models A)$, for another condition must be added to deal with negation. This is dealt with in the following definition and theorem.

Definition 5.8 Given an interpretation $\langle g, W, R, \models, * \rangle$, if binary relation $\sqsubseteq$ on W satisfies

$$a \sqsubseteq b \Rightarrow \begin{cases} a \models p \Rightarrow b \models p & \text{for every propositional variable } p, \\ Rcbd \Rightarrow Rcad & \text{if } a \neq g, \\ Rcbd \Rightarrow c \sqsubseteq d & \text{if } a = g, \\ b^* \sqsubseteq a^*, & \end{cases}$$

it is said to be a *containment* relation on $\langle g, W, R, \models, * \rangle$.

THEOREM 5.21 *Let $\langle g, W, R, \models, * \rangle$ be an interpretation, and let $\sqsubseteq$ be a containment relation on W. Then $a \sqsubseteq b \Rightarrow (a \models A \Rightarrow b \models A)$ for every formula A.*

Proof: We add a clause for $\sim$ to the induction on the complexity of formulae. If $a \sqsubseteq b$ and the result holds for A, then if $a \models {\sim} A$ it follows that $a^* \not\models A$, and as $b^* \sqsubseteq a^*$ it must be that $b^* \not\models A$ and hence that $b \models {\sim} A$ as desired. ◁

The extension results in the previous sections carry over to the logic **B** with no modification. What we are interested in is the possibility of extending **B** with axioms or rules that use negation. This way we can model logics like **DW** which validate more negation principles, like contraposition in axiom form. This can be done, as the following theorem shows.

THEOREM 5.22 *For each row in Table 5.3 the logic* **B** *with an axiom added is sound and complete with respect to the class of* **B** *interpretations $\langle g, W, R, \models, * \rangle$ where R satisfies the corresponding condition, and for the last axiom, the interpretations are assumed to be extended with a containment relation $\sqsubseteq$.*

Red	$(A \to {\sim} A) \to {\sim} A$	Ra^*aa for $a \neq g$, and $g^* \sqsubseteq g$
CPA	$(A \to {\sim} B) \to (B \to {\sim} A)$	$Rabc \Rightarrow Rc^*ba^*$
ExMid	$A \vee {\sim} A$	$g^* \sqsubseteq g$

Table 5.3

Proof: These are proved in exactly the same way as the other extensions.

Red. Assume that Raa^*a for each a.

$$
\begin{array}{l}
1 \quad g \not\models A \to (A \to A) \\
\overbrace{\hspace{12em}}
\end{array}
$$

$$
\begin{array}{llc|llc}
2 & x \neq g & [1] & 5 & g \models A \to \sim A & [1] \\
3 & x \models A \to \sim A & [1] & 6 & g \not\models \sim A & [1] \\
4 & x \not\models \sim A & [1] & 10 & g^* \models A & [6] \\
7 & x^* \models A & [4] & 11 & g^* \sqsubseteq g & [\text{Red}] \\
8 & Rx^*xx & [2,\text{Red}] & 12 & g \models A & [10,11] \\
9 & x \models \sim A & [3,7,8] & 13 & g \models \sim A & [5,12] \\
 & \times & [4,9] & & \times & [6,13]
\end{array}
$$

Now assume that **Red** holds and that Σ is a prime Π-theory, distinct from Π. We wish to show that $R\Sigma^*\Sigma\Sigma$, so let $A \to B \in \Sigma$ and $A \in \Sigma^*$. The thing to note is that if **Red** holds, so must $(A \to B) \to (\sim A \vee B)$. To see this, consider the following derivation:

$\vdash_\Pi A \wedge \sim B \to A, \quad \vdash_\Pi B \to \sim A \vee B,$	
$\vdash_\Pi \sim A \vee B \to \sim(A \wedge \sim B),$	by **DM2** and **DN**,
$\vdash_\Pi B \to \sim(A \wedge \sim B),$	by transitivity,
$\vdash_\Pi (A \to B) \to ((A \wedge \sim B) \to \sim(A \wedge \sim B)),$	by prefixing and suffixing,
$\vdash_\Pi ((A \wedge \sim B) \to \sim(A \wedge \sim B)) \to \sim(A \wedge \sim B),$	by **Red**,
$\vdash_\Pi (A \to B) \to \sim(A \wedge \sim B),$	by transitivity,
$\vdash_\Pi \sim(A \vee \sim B) \to (\sim A \vee B),$	by **DM2**, **DN**,
$\vdash_\Pi (A \to B) \to (\sim A \vee B),$	by transitivity.

So we have $\sim A \vee B \in \Sigma$, and $\sim A \notin \Sigma$, giving $B \in \Sigma$, as we wanted.

To show that $\Pi^* \subseteq \Pi$, note that $A \vee \sim A$ is a theorem. Take $A \in \Pi^*$, so $\sim A \notin \Pi$. But $A \vee \sim A \in \Pi$, so $A \in \Pi$ as desired.

CPA. Assume that $Rabc \Rightarrow Rc^*ba^*$

$$
\begin{array}{lcc}
1 & g \not\models (A \to B) \to (\sim B \to \sim A) & \\
2 & x \models A \to B & [1] \\
3 & x \not\models \sim B \to \sim A & [1] \\
4 & Ryxz & [3] \\
5 & y \models \sim B & [3] \\
6 & z \not\models \sim A & [3] \\
7 & z^* \models A & [6] \\
8 & Rz^*xy^* & [4,\text{CPA}] \\
9 & y^* \models B & [2,7,8] \\
10 & y \not\models \sim B & [9] \\
 & \times & [3,9]
\end{array}
$$

Now assume that **CPA** holds and that Σ, Γ and Δ are prime Π theories such that $R\Sigma\Gamma\Delta$. Let $A \to B \in \Gamma$ and $A \in \Delta^*$, i.e., $\sim A \notin \Delta$. By **CPA** we must have $\sim B \to \sim A \in \Gamma$, so we must have $\sim B \notin \Sigma$, lest $\sim A \in \Delta$. This then gives $B \in \Sigma^*$, which ensures that $R\Delta^*\Gamma\Sigma^*$. (If $\Gamma = \Pi$, the result is even easier to prove.)

ExMid. Assume that $g^* \sqsubseteq g$.

$$
\begin{array}{lll}
1 & g \not\models A \vee {\sim} A & \\
2 & \quad g \not\models A & [1] \\
3 & \quad g \not\models {\sim} A & [1] \\
4 & \quad g^* \models A & [3] \\
5 & \quad g \models A & [4,\text{ExMid}] \\
 & \qquad \times & [2,5]
\end{array}
$$

Now assume that **ExMid** holds. We want to show that $\Pi^* \subseteq \Pi$ — to this end, note that if $A \in \Pi^*$, ${\sim} A \notin \Pi$ and so $A \vee {\sim} A \in \Pi$ ensures that $A \in \Pi$ as desired. ◁

5.8 The Logics we have Covered

It is time to take stock and consider what logics have a semantics as the result of these investigations. It is clear that we have covered $\mathbf{DW}^{d+}$, $\mathbf{BM}^{d}$ and $\mathbf{B}^{d}$, and any logic that can be obtained by adding the axioms we considered.

Firstly, we have modelled our favourite logics (or some close cousins of them), $\mathbf{DW}^{d} = \mathbf{B}^{d} + \mathbf{CPA}$, $\mathbf{TW}^{d} = \mathbf{DW}^{d} + \mathbf{B} + \mathbf{B}'$, $\mathbf{EW}^{d} = \mathbf{TW} + \mathbf{C}''$, $\mathbf{C} = \mathbf{DW}^{d} + \mathbf{C}$, $\mathbf{CK} = \mathbf{C} + \mathbf{K}$. We have also modelled some logics given by extending our favourites with some more untoward principles. For example, $\mathbf{E}^{d} = \mathbf{EW} + \mathbf{W}$, $\mathbf{R} = \mathbf{C} + \mathbf{W}$, and $\mathbf{K} = \mathbf{CK} + \mathbf{W}$.

5.9 Boolean Negation

As a formal construction, it is possible to add to these logics a 'negation' commonly called "Boolean Negation," which we will write as '$-$.' It is characterised by the following axioms. (See Giambrone and Meyer's "Completeness and Conservative Extension Results for Some Boolean Relevant Logics" [49] for this characterisation.)

$$A \to (B \to C \vee -C) \qquad \mathbf{BA1}$$

$$-(A \to B) \vee (-A \vee B) \qquad \mathbf{BA2}$$

$$A \wedge -A \to B \qquad \mathbf{BA3}$$

If a logic $\mathbf{X}$ is without Boolean negation, the logic resulting from adding such a negation is called '$\mathbf{CX}$.' It is well-known that Boolean negation satisfies $\vdash --A \leftrightarrow A$, $\vdash A \wedge -A \to B$ and $\vdash A \to B \vee -B$, and I will not prove that here. To model Boolean negation in the simplified semantics, we add the obvious condition that:

$$w \models -A \iff w \not\models A$$

It is trivial to show that the semantics for $\mathbf{X}$ with this extension is sound and complete for $\mathbf{CX}$, using well-known properties of Boolean negation. However, this gives us a conservative extension result, which is a corollary of the following lemma.

THEOREM 5.23 *Given any* $\mathbf{BM}$ *or* $\mathbf{DW}^{+}$ *interpretation, not using a containment relation, the structure given by adding the rule for Boolean negation has exactly the same evaluation as the original on formulae that do not contain '$-$.'*

Proof: By inspection. The reason a containment relation is not permitted is that the hereditary condition on the relation fails in general, given the presence of '$-$.' (See endnote 3). ◁

As a simple corollary we have:

COROLLARY *If* **X** *is a logic which has a sound and complete simplified semantics, not using a containment relation, then* **CX** *is a conservative extension of* **X**.

It follows that **CR**, **CC**, $\mathbf{CTW^d}$, $\mathbf{CDW^d}$, $\mathbf{CEW^d}$ and $\mathbf{CE^d}$ are conservative extensions of **R**, **C**, $\mathbf{TW^d}$, $\mathbf{DW^d}$ and $\mathbf{EW^d}$ and $\mathbf{E^d}$ respectively — and other less known logics are also conservatively extended. The results for **R**, **C**, **TW** and **E** were known, but those for **DW** and $\mathbf{EW^d}$ are new.

Other logics such as **CCK** are not proved to conservatively extend **CK** — as their semantics use the inclusion relation. For **CK**, there is a good reason why the extension result cannot be proved.

THEOREM 5.24 **CCK** *is not a conservative extension of* **CK**

Proof: Firstly, $A \vee {\sim} A$ is not a theorem of **CK**. We will show that it is a theorem of **CCK**. To do this, note that $A \vee -A$ holds in **CCK**, so it is enough to show that $-A \to {\sim} A$.

In **CK**, it is simple to show that $\vdash {\sim} A \leftrightarrow (A \to \bot)$ for some contradiction '$\bot$.' For example, $\vdash {\sim} A \leftrightarrow (A \to {\sim}(A \to A))$ in **CK**, and hence in **CCK**. Now we have $\vdash A \to (-A \to A \wedge -A)$ and $\vdash A \wedge -A \to {\sim}(-A \to -A)$, so we have $\vdash A \to (-A \to {\sim}(-A \to -A))$. But this gives $\vdash A \to {\sim} -A$, which contraposed is $\vdash -A \to {\sim} A$, as desired. ◁

5.10 Validities

It is simple to model world validity in these structures. Weak validity is modelled as preservation at g in all structures. The index g does the job of 'logic'. Strong validity is modelled as preservation at all indices in the structures. (As a conditional is true at g (logically true) just when it is truth preserving at all indices.) The way is open to define world validity in our structures too. This can be defined as truth preservation at a limited class of indices, which do the job of worlds. These will satisfy a number of conditions, as they have to be sufficiently worldlike. Taking worlds to validate logic, and to be detached, we would like $g \sqsubseteq w$ and $Rwww$ for each world w. For excluded middle for worlds (completeness) we require $w^* \sqsubseteq w$, and for consistency, we want $w \sqsubseteq w^*$. Proving soundness and completeness for these notions is a trivial exercise. The methods of these chapters will suffice. We must leave the fleshing out of the details for another occasion, and instead, move on to consider an alternative approach to negation.

5.11 Notes

An earlier version of this chapter appeared in the *Journal of Philosophical Logic* [124]. I would like to thank the anonymous referee, who rescued me from a number of errors, and gave suggestions pointing to the Boolean negation results.

[1]This is *obvious* when you think about it. Take the logic **C** for example. Define another logic **C**′ has as axioms the theorems of **C**, but has *no* rules. Then it has the strongly valid arguments of **C** (as these are read off the theorems: $X \Vdash A$ if and only if $\vdash \mathcal{I}(X) \to A$) while having very few weakly valid arguments. (The argument from Σ to A is weakly valid if and only if A is either an axiom or in A.)

[2]We're deviating from the standard notation here, in that $Rxyz$ means that x and z are related together by y (or are compossible with respect to y). This is to follow our suggestive notation that has $\vdash A \circ (A \to B) \to B$, and so on. We follow Dunn's later work [35], instead of his earlier work [30] and almost everything and everyone else.

[3]If Boolean negation is present, it can be used to *show* that $R\Gamma\Pi\Delta$ if and only if $\Gamma = \Delta$ (given that Γ is non-empty and Δ is not full). It is quite simple to do: Boolean negation (written as '$-$') satisfies $\vdash A \to B \vee -B$ and $\vdash A \wedge -A \to B$. It follows that for any non-empty, non-full Π-theories Γ, $A \vee -A \in \Gamma$, and $A \wedge -A \notin \Gamma$, so exactly *one* of A and $-A$ are in Γ. If $R\Gamma\Pi\Delta$, then it is clear that $\Gamma \subseteq \Delta$ (as Π contains all identities), and so, as Γ is non-empty, it contains exactly one element of each $\{A, -A\}$ pair, for each A. So, Δ contains at least this element for each pair. However, it cannot contain *both*, for any pair (being non-full), so Δ contains exactly the same elements of each pair as does Γ. Hence, $\Gamma = \Delta$. The other direction of the biconditional is obvious, given that Γ and Δ are Π-theories.

The behaviour of Boolean negation is important, when we come to the last section, where we show that Boolean negation conservatively extends a large class of logics.

[4]For completeness' sake, the candidates given by Routley [134] that seem to need non-degeneracy, but for which the current results will not hold (and Routley's proofs do not seem to work) are $A \vee (A \to B)$ (Routley's B16), $(A \to B) \to ((A \to C) \to (A \to B \wedge C))$ (B14) and $A \vee B \to ((A \to B) \to B)$ (B19).

Chapter 6
Simplified Semantics 2

When we are young we are a jungle of complications.
We simplify as we get older.
— GRAHAM GREENE *The Quiet American* [53]

We will learn in Chapters 8 and 9 that there are plenty of uses for a ternary relational semantics. However, none of them find a decent interpretation for the dualising operator $*$. So, it is important to consider different possibilities for modelling negation. That is the task for this chapter.

6.1 Definitions

In their 'Simplified Semantics for Basic Relevant Logics' [118], Priest and Sylvan also used a four-valued evaluation to model some weak logics. Their results deal with **BD**, a weak relevant propositional logic that extends $\mathbf{DW}^+$ by adding a negation satisfying the three axioms:

$$\sim(A \vee B) \leftrightarrow \sim A \wedge \sim B \qquad \mathbf{DM1}$$

$$\sim(A \wedge B) \leftrightarrow \sim A \vee \sim B \qquad \mathbf{DM2}$$

$$A \leftrightarrow \sim\sim A \qquad \mathbf{DN}$$

So, **BD** licences intuitionistically unacceptable principles such as double negation elimination, but it doesn't validate some rather basic principles, such as any form of contraposition.

Their semantics is defined as follows: an *interpretation* for the language is a 5-tuple $\langle g, W, R, \models, =\!\!|\rangle$, where W is a set of indices, $g \in W$ is the base index, R is a ternary relation on W such that $Rxgy$ if and only if $x = y$, and $\models$ and $=\!\!|$ are binary relations between indices and formulae. The relations are restricted to satisfying the following conditions:

- $w \models A \wedge B \iff w \models A$ and $w \models B$,
- $w =\!\!| A \wedge B \iff w =\!\!| A$ or $w =\!\!| B$,
- $w \models A \vee B \iff w \models A$ or $w \models B$,
- $w =\!\!| A \wedge B \iff w =\!\!| A$ and $w =\!\!| B$,
- $w \models \sim A \iff w =\!\!| A$,
- $w =\!\!| \sim A \iff w \models A$,
- $w \models A \to B \iff$ for all $y, z \in W$ $(Rywz \Rightarrow (y \models A \Rightarrow z \models B))$,
- $w \models A \circ B \iff$ for some $y, z \in W$, $Rxyw$, $x \models A$ and $y \models B$.

This evaluation is now explicitly four-valued at each world. Any proposition can be supported at a world, undermined at a world, both, or neither. We are up-front about this in the evaluation. This makes the four-valued approach to ternary relational semantics part of the tradition started by Dunn in his semantics for first-degree entailment [29]. The $*$ operator kept this feature hidden, but in the $*$ semantics, there are also four possibilities for the truth and falsity of formulae at an index. Either A is true at w and not at w^* (so $\sim A$ is not true at w) or A is not true at w and true at w^* (so $\sim A$ is true at w) or A is true at both w and w^* (so both A and $\sim A$ are true at w) or A is not true at either w or w^* (so neither of A or $\sim A$ are true at w). The four-valued interpretation eschews all mention of $*$, and evaluates formulae in terms of their being supported or undermined at indices.

Note that the *falsity* of conditionals and fusions are arbitrary according to this definition of an evaluation. The trouble that this causes will be made obvious later.

Then weak validity is defined in terms of truth preservation at g, the base index. In other words,

$$\Theta \vDash \mathbf{A} \iff \text{ for all interpretations } \langle g, W, R, \models, \dashv \rangle \ (g \models B \text{ for all } B \in \Theta \Rightarrow g \vDash A).$$

The soundness and completeness result for **BD** can then be concisely stated as follows.

Theorem 6.1 *If $\Theta \cup \{A\}$ is a set of formulae, then*

$$\Theta \vdash A \iff \Theta \vDash A,$$

where $\vdash$ is weak validity in **BD**.

The proof here is totally standard, and we omit it. Some of the *positive* extensions of this semantics in Table 5.1 trivially extend to the four-valued interpretation — the proofs in the previous chapter are proofs in the four-valued context.

In that chapter, we defined the notion of a containment relation on an interpretation to deal with more extensions to the basic logics (those in Table 5.2). To have an analogous relation in the four-valued context, we need both that $a \sqsubseteq b \Rightarrow (a \models A \ \Rightarrow \ b \models A)$, and $a \sqsubseteq b \Rightarrow (a \dashv A \ \Rightarrow \ b \dashv A)$, to ensure that the induction step for negation goes through. Unfortunately, as the falsity condition for a conditional is arbitrary, there is no way of ensuring that the latter condition is satisfied in the case where A is a conditional. So, the extensions that need a containment relation cannot be modelled in this way. Extensions involving negation are likewise ruled out, for they explicitly use the dualising '$*$' operator, which is unavailable here. So, we are left in a sorry state. There are two alternatives available. Firstly, we will unashamedly steal from the $*$-semantics to give a four-valued interpretation for (almost) anything that has a $*$-interpretation, and secondly, we will add negation conditions for implication, to give a smoother semantics for a number of systems.

6.2 Theft

The matter of finding a four-valued semantics for **DW** or its weaker cousin **B** is not as simple as giving it a $*$-semantics. We will show that such a semantics does exist, and demonstrate soundness and completeness. First, however, we need a preliminary result, and some terminology. A two-valued interpretation with a '$*$' operator will henceforth be called a *$*$-interpretation*. The first result establishes the connection between these $*$-interpretations and four-valued interpretations.[1]

THEOREM 6.2 *For any $*$-interpretation that models* **B** *there is a four-valued interpretation on the same set of worlds, with exactly the same truths in each world.*

Proof: Let the $*$-interpretation be $\langle g, W, R, \models_*, * \rangle$. We define a four-valued interpretation $\langle g, w, R, \models_4, =\!\!|_4 \rangle$ by requiring

$$w \models_4 p \text{ if and only if } w \models_* p$$
$$w =\!\!|_4 p \text{ if and only if } w^* \not\models_* p$$

for each world w and atomic formula p, where $*$ is the world 'inversion' map of the $*$-system.

We will show that the four-valued interpretation has exactly the same truths in each world as the $*$-interpretation.

$$\begin{aligned}
w \models_4 A \wedge B &\iff w \models_4 A \text{ and } w \models_4 B \\
&\iff w \models_* A \text{ and } w \models_* B \\
&\iff w \models_* A \wedge B \\
w =\!\!|_4 A \wedge B &\iff w =\!\!|_4 A \text{ or } w =\!\!|_4 B \\
&\iff w^* \not\models_* A \text{ or } w^* \not\models_* B \\
&\iff w^* \not\models_* A \wedge B
\end{aligned}$$

$$\begin{aligned}
w \models_4 A \vee B &\iff w \models_4 A \text{ or } w \models_4 B \\
&\iff w \models_* A \text{ or } w \models_* B \\
&\iff w \models_* A \vee B \\
w =\!\!|_4 A \vee B &\iff w =\!\!|_4 A \text{ and } w =\!\!|_4 B \\
&\iff w^* \not\models_* A \text{ and } w^* \not\models_* B \\
&\iff w^* \not\models_* A \vee B \\
w \models_4 {\sim} A &\iff w =\!\!|_4 A \\
&\iff w^* \not\models_* A \\
&\iff w \models_* {\sim} A \\
w =\!\!|_4 {\sim} A &\iff w \models_4 A \\
&\iff w \models_* A \\
&\iff w^* \not\models_* {\sim} A \\
w \models_4 A \to B &\iff \forall x, y (Rxwy \Rightarrow x \models_4 A \;\Rightarrow\; y \models_4 B) \\
&\iff \forall x, y (Rxwy \Rightarrow x \models_* A \;\Rightarrow\; y \models_* B) \\
&\iff w \models_* A \to B \\
w \models_4 A \circ B &\iff \exists x, y (Rxyw,\ x \models_4 A,\ y \models_4 B) \\
&\iff \exists x, y (Rxyw,\ x \models_* A,\ y \models_* B) \\
&\iff w \models_* A \circ B
\end{aligned}$$

So, every $*$-interpretation that models **B** gives a four-valued interpretation on the same set of worlds and with the same truths in the same worlds. ◁

We say that a four-valued interpretation with a corresponding $*$-interpretation is *closed under duality*, because it is easy to show that a four-valued interpretation has a corresponding $*$-interpretation if and only if for every world w, there is another world w^* such that

$$w \models A \text{ and } w \not\dashv A \iff w^* \not\dashv A \text{ and } w^* \models A$$
$$w \not\models A \text{ and } w \dashv A \iff w^* \dashv A \text{ and } w^* \not\models A$$
$$w \models A \text{ and } w \dashv A \iff w^* \not\dashv A \text{ and } w^* \not\models A$$
$$w \not\models A \text{ and } w \not\dashv A \iff w^* \dashv A \text{ and } w^* \models A$$

for each sentence A. The world w^* is said to be the *dual* of the world w. The presence of duals for each world is enough to model the * operator from a $*$-interpretation. A four-valued interpretation is closed under duality if and only if the dual of every world in the interpretation is also a world in the interpretation. This condition is too difficult to check when presented with a four-valued model. An equivalent condition that is simpler to check is provided in the following theorem.

THEOREM 6.3 *A four-valued interpretation* $\langle g, W, R, \models, \dashv \rangle$ *that models* **B** *is closed under duality if and only if there is an involution* $* : W \to W$ *such that*

$$w \dashv p \textit{ if and only if } w^* \not\models p$$
$$w \dashv A \to B \textit{ if and only if } w^* \not\models A \to B$$

for each world w*, propositional parameter* p*, and formulae* A *and* B*.*

Proof: The 'only if' part is immediate, as these conditions are a subset of the conditions we have closure under duality. The 'if' part is a matter of proving that $w \dashv A$ if and only if $w^* \not\models A$ for each w and A.

To show that $w \dashv A$ if and only if $w^* \not\models A$, we use induction on the complexity of formulae. The base case, and the case for $\to$ are given. The case for conjunction is as follows: $w \dashv A \wedge B$ if and only if $w \dashv A$ or $w \dashv B$ if and only if $w^* \not\models A$ or $w^* \not\models B$ if and only if $w^* \not\models (A \wedge B)$. The cases for disjunction and negation are similar, and are left as an exercise. ◁

This condition is simple enough to check at the level of propositional parameters, and the remaining portion of the condition deals with falsity conditions for entailments, which have to be explicitly specified in a four-valued model, in any case. While this is saving the four-valued interpretation by an explicit use of '$*$,' which the four-valued interpretation is designed to avoid, there does not seem to be any way of avoiding it, if the truth conditions of entailments are to be kept as they are, as some kind of duality operator is the natural way to model rule-contraposition, which is the characteristic rule of **B**. In any case the construction we have just given is enough to prove the following theorem.

THEOREM 6.4 **B** *is sound and complete with respect to the collection of four-valued interpretations closed under duality.*

Proof: It is an immediate corollary of the fact that **B** is sound and complete with respect to the $*$-interpretations that satisfy $w^{**} = w$, but an independent proof is possible. We need to show that **CPR** holds in all four-valued models closed under duality. To see that this is the case, assume that for all w, $w \models A \Rightarrow w \models B$. So, if $w \models \sim B$, we must have $w \dashv B$, and so, $w^* \not\models B$. This means that $w^* \not\models A$ by our assumption, and hence $w \dashv A$. This results in $w \models \sim A$, which is what we wanted. This gives us soundness.

Completeness can be obtained by the standard canonical model construction; it suffices to show that in the interpretation consisting of the prime Π-theories, every world has a dual. The choice of $\Sigma^* = \{A : \sim A \notin \Sigma\}$ works. It is simple to show that this is satisfies the conditions for duality, under the assumption of **CPR**. ◁

In this way, any four-valued model of **B** can be converted to a two-valued model, and conversely. So the results for axioms and rules extending **B** also hold for the four-valued semantics, when an appropriate dualising operation $*$ is made explicit. This means that we can use the former definition of the containment relation:

$$a \sqsubseteq b \Rightarrow \begin{cases} a \models p \;\Rightarrow\; b \models p & \text{for every propositional variable } p, \\ Rcbd \Rightarrow Rcad & \text{if } a \neq g, \\ Rcbd \Rightarrow c \sqsubseteq d & \text{if } a = g. \\ b^* \sqsubseteq a^* & \end{cases}$$

and it will satisfy the condition that for all A, if $a \sqsubseteq b$, then $a \models A \Rightarrow a \models B$. So, the results of the previous chapter show us that we can extend **B** by adding any axiom from among those in Table 5.2, and the semantics resulting from adding the corresponding modelling condition is sound and complete with respect to it.

The four-valued semantics given here is not particularly exciting — the reason for the four-valued semantics is to get *away* from dualising operators, and to give negation a more pleasing modelling. One way of doing this is to introduce *another* ternary relation S, to deal with false conditionals. This is the original American plan, and could be followed in the simplified case. However, there is a smoother possibility that uses neither $*$ nor a ternary S, and which will capture the relevant logic **C** (but not much else, it seems).

6.3 Natural Four-Valued Semantics

Priest and Sylvan note [118] that the thing that makes the four-valued semantics difficult is contraposition. One way to address this is to rewrite the truth conditions for the conditional as follows, 'wiring in' the validity of contraposition:

$$\begin{aligned} &w \models A \to B \text{ if and only if for each } x, y \text{ where } Rxwy, \\ &\qquad\qquad \text{if } x \models A \text{ then } y \models B \text{ and if } x \dashv B \text{ then } y \dashv A. \end{aligned}$$

In this case, the rule form of contraposition (**CPR**) follows, but some of the theorems and rules of $\mathbf{B}^+$ fail. For example, $\wedge$-introduction:

$$(A \to B) \wedge (A \to C) \to (A \to B \wedge C)$$

is falsified in a structure with a world w where

$$w \Dashv (A \to B \land C) \text{ and } w \not\Dashv (A \to B) \land (A \to C)$$

and as the falsity conditions of conditionals is purely arbitrary, such structures exist. To make sense of these axioms in the context of an evaluation which validates contraposition, we need a policy for the falsity of conditionals. One policy for the falsity of conditionals is motivated by the result from strong relevant logics, to the effect that where $A \circ B$ (the 'fusion' of A and B) is defined as $\sim(A \to \sim B)$, we have that:

$$\vdash (A \circ B \to C) \leftrightarrow (B \to (A \to C))$$

This gives a connection between a negated conditional — $A \circ B$ — and a purely positive formula. The corresponding condition on the conditional would then be:

$$w \Dashv A \to B \text{ if and only if there are } x, y \text{ where } Rxyw,\ x \models A \text{ and } y \Dashv B$$

And the falsity condition for fusion becomes:

$$\begin{aligned} w \Dashv A \circ B \text{ if and only if for each } x, y \text{ where } Rxwy, \\ \text{if } x \models A \text{ then } y \Dashv B \text{ and if } x \models B \text{ then } y \Dashv A. \end{aligned}$$

Given this, the contraposition *axiom* becomes logically true, so the semantics is sound for **DW** but is not complete. If we add the condition **CI** (recall Table 5.1) that $Rabc \Rightarrow Rbac$ for each a, b, c, we have completeness for **DW+CI**, which we will call **DWCI**. It is important that **CI** hold, for it is only in its presence that the biconditional connecting the fusion and the nested implication. We call this kind of semantics a *natural four-valued semantics*.

THEOREM 6.5 *The natural four-valued semantics is sound with respect to* **DWCI**.

Proof: We need to show that if $\langle g, W, R, \models, \Dashv \rangle$ is an interpretation, then the axioms of **DWCI** hold at g, and the rules are truth-preserving at g. It is not *entirely* trivial because of the double entry bookkeeping of four-valuedness, so we will work the details for *disjunction elimination*, *modus ponens*, *transitivity*, **CI** and one of the fusion rules, and leave the rest for the reader. The first is large, and it's split into three parts.

$$\begin{array}{l}
{}_1 \quad g \not\models (A \to C) \land (B \to C) \to (A \lor B \to C) \\
\begin{array}{ll|ll}
{}_2\ w \models (A \to C) \land (B \to C) & {}_{[1]} & {}_4\ w \Dashv A \lor B \to C & {}_{[1]} \\
{}_3\ w \not\models A \lor B \to C & {}_{[1]} & {}_5\ w \not\Dashv (A \to C) \land (B \to C) & {}_{[1]}
\end{array}
\end{array}$$

The left branch is this:

$$
\begin{array}{lcr}
2 & w \models (A \to B) \land (B \to C) & [1] \\
3 & w \not\models A \lor B \to C & [1]
\end{array}
$$

$$
\begin{array}{lcr|lcr}
6 & x \models A \lor B & [3] & 9 & x \Dashv C & [3] \\
7 & y \not\models C & [3] & 10 & y \not\Dashv A \lor B & [3] \\
8 & Rxwy & [3] & 11 & Rxwy & [3]
\end{array}
$$

$$
\begin{array}{lcr|lcr|lcr|lcr}
12 & x \models A & [6] & 13 & x \models B & [6] & 18 & y \not\Dashv A & [10] & 19 & y \not\Dashv B & [10] \\
14 & w \models A \to C & [2] & 16 & w \models B \to C & [2] & 20 & w \models A \to C & [2] & 22 & w \models B \to C & [2] \\
15 & y \models C & [8,12,14] & 17 & y \models C & [8,13,16] & 21 & x \not\Dashv C & [11,18,20] & 23 & x \not\Dashv C & [10,19,22] \\
 & \times & [7,15] & & \times & [7,17] & & \times & [9,21] & & \times & [9,23]
\end{array}
$$

and the right branch is this:

$$
\begin{array}{lcr}
4 & w \Dashv A \lor B \to C & [1] \\
5 & w \not\Dashv (A \to B) \land (B \to C) & [1] \\
24 & w \not\Dashv A \to C & [5] \\
25 & w \not\Dashv B \to C & [5] \\
26 & Rxyw & [4] \\
27 & x \models A \lor B & [4] \\
28 & y \Dashv C & [4]
\end{array}
$$

$$
\begin{array}{lcr|lcr}
29 & x \models A & [26,28,29] & 30 & x \models B & [27] \\
31 & w \not\Dashv A \to C & [26,28,29] & 32 & w \not\Dashv B \to C & [36,28,30] \\
 & \times & [24,31] & & \times & [25,37]
\end{array}
$$

For rules we need only validate the non-disjunctive forms.

$$
\begin{array}{lc}
1 & g \models A \to B \\
2 & g \models C \to D \\
3 & g \not\models (B \to C) \to (A \to D)
\end{array}
$$

$$
\begin{array}{lcr|lcr}
4 & x \models B \to C & [3] & 6 & x \Dashv A \to D & [3] \\
5 & x \not\models A \to D & [3] & 7 & x \not\Dashv B \to C & [3] \\
8 & Ryxz & [5] & 14 & Ryzx & [6] \\
9 & y \models A & [5] & 15 & y \models A & [6] \\
10 & z \not\models D & [5] & 16 & z \Dashv D & [6] \\
11 & y \models B & [1,9] & 17 & y \models B & [1,15] \\
12 & z \models C & [4,8,11] & 18 & z \Dashv C & [2,16] \\
13 & z \models D & [2,12] & 19 & x \Dashv B \to C & [14,18,17] \\
 & \times & [10,13] & & \times & [7,19]
\end{array}
$$

Assertion is simple:

1	$g \not\models A \to ((A \to B) \to B)$							
2	$w \models A$	[1]				4	$w \Dashv (A \to B) \to B$	[1]
3	$w \not\models (A \to B) \to B$	[1]				5	$w \not\Dashv A$	[1]
						16	$Rxyw$	[4]
6	$Rxwy$	[3]	9	$Rxwy$	[3]	17	$x \models A \to B$	[4]
7	$x \models A \to B$	[3]	10	$x \Dashv B$	[3]	18	$y \Dashv B$	[4]
8	$y \not\models B$	[3]	11	$y \not\Dashv A \to B$	[3]	19	$Ryxw$	[16,CI]
12	$Rwxy$	[6,CI]	14	$Rwxy$	[9,CI]	20	$w \Dashv A$	[17,18,19]
13	$y \models B$	[2,7,12]	15	$y \Dashv A \to B$	[2,11,14]		$\times$	[5,20]
	$\times$	[8,13]		$\times$	[11,15]			

Finally, for fusion we have

1	$g \models A \circ B \to C$							
2	$g \not\models B \to (A \to C)$							
3	$w \models B$	[2]				5	$w \Dashv A \to C$	[2]
4	$w \not\Dashv A \to B$	[2]				6	$w \not\Dashv B$	[2]
						18	$Rxyw$	[5]
7	$Rxwy$	[4]	10	$Rxwy$	[4]	19	$x \models A$	[5]
8	$x \models A$	[4]	11	$x \Dashv C$	[4]	20	$y \Dashv C$	[5]
9	$y \not\models C$	[4]	12	$y \not\Dashv A$	[4]	21	$y \Dashv A \circ B$	[1,20]
13	$y \models A \circ B$	[3,7,18]	15	$x \Dashv A \circ B$	[1,11]	22	$w \Dashv B$	[18,19,21]
14	$y \models C$	[1,13]	16	$Rwxy$	[10,CI]		$\times$	[6,22]
	$\times$	[9,14]	17	$y \Dashv A$	[3,15,16]			
				$\times$	[12,17]			

◁

THEOREM 6.6 *The natural four-valued semantics is complete with respect to* **DWCI**.

Proof: As in the previous chapter, we will use the almost canonical model, modified to satisfy permutation. In other words, we will show that the collection of all prime Π-theories (with Π duplicated in the usual fashion), with R defined by demanding $R\Sigma\Gamma\Delta$ if and only if $\forall A, B,\ A \to B \in \Gamma \Rightarrow (A \in \Sigma \Rightarrow B \in \Delta)$ for $\Sigma, \Gamma \neq \Pi$, and $R\Pi\Gamma\Delta$ if and only if $R\Gamma\Pi\Delta$ if and only if $\Gamma = \Delta$.

Note that if $R\Sigma\Gamma\Delta$, and $A \to B \in \Sigma$, then $\sim B \to \sim A \in \Sigma$ by contraposition, and hence, if $\sim B \in \Gamma$, $\sim A \in \Delta$, so our definition parallels the truth-conditions for conditionals, albeit, not explicitly.

Define $\models$ and $\Dashv$ by setting $\Sigma \models p$ iff $p \in \Sigma$ and $\Sigma \Dashv p$ iff $\sim p \in \Sigma$, and let it satisfy the usual inductive definitions of an interpretation. We just need to show that $\Sigma \models A$ if and only if $A \in \Sigma$, and $\Sigma \Dashv A$ if and only if $\sim A \in \Sigma$ for *every* formula A. We do this by the usual induction on the complexity of the formulae. The only difficult case is that for implication, so we leave the rest for the reader.

Firstly, $\Sigma \models A \to B$ if and only if for each Γ, Δ where $R\Gamma\Sigma\Delta$ $(\Gamma \models A \Rightarrow \Delta \models B)$. If $A \to B \in \Sigma$, then we have for each Γ, Δ where $R\Gamma\Sigma\Delta$ $(\Gamma \models A \Rightarrow \Delta \models B)$ by the definition of R, given the induction hypothesis.

Conversely, if $A \to B \notin \Sigma$, then we have Γ, Δ where $R\Gamma\Sigma\Delta$, $A \in \Gamma$ and $B \notin \Delta$, (even in the case $\Sigma = \Pi$) by Lemma 5.8. To show that $R\Sigma\Gamma\Delta$, note that if $\Gamma = \Pi$, we can take $\Gamma = \Pi'$.

For the other case, $\Sigma \mathrel{=\!\!|} A \to B$ if and only if there are Γ and Δ where $R\Gamma\Delta\Sigma$, $A \in \Gamma$ and ${\sim}B \in \Delta$.

Firstly, if $R\Gamma\Delta\Sigma$, $A \in \Gamma$ and ${\sim}B \in \Delta$, then we have that $(A \to B) \to B \in \Gamma$ by **CI**, which contraposed gives ${\sim}B \to {\sim}(A \to B) \in \Gamma$, so $R\Gamma\Delta\Sigma$ gives ${\sim}(A \to B) \in \Sigma$.

Conversely, if ${\sim}(A \to B) \in \Sigma$, set $\Gamma = \{C : \vdash_\Pi A \to C\}$ and $\Delta = \{C : \vdash_\Pi {\sim}B \to C\}$. Then $A \in \Gamma$ and ${\sim}B \in \Delta$. To show that $R\Gamma\Delta\Sigma$, assume that $C \in \Gamma$ and $C \to D \in \Delta$. Then $\vdash_\Pi A \to C$ and $\vdash_\Pi {\sim}B \to (C \to D)$. Then $\vdash_\Pi {\sim}D \to (C \to B)$ (permuting and contraposing) and $\vdash_\Pi (C \to B) \to (A \to B)$ (by transitivity) give $\vdash_\Pi {\sim}D \to (A \to B)$ which then gives $\vdash_\Pi {\sim}(A \to B) \to D$, and so, $D \in \Sigma$. Applying Lemma 5.11 this gives prime theories $\Gamma' \supseteq \Gamma$ and $\Delta' \supseteq \Delta$ where $R\Gamma'\Delta'\Sigma$. To ensure that $R\Gamma'\Delta'\Sigma$, if either of Γ' or Δ' are equal to Π when considered as sets, identify them with Π'. Cases for fusion are similar, and are left as an exercise. This gives us the result, and completes the proof. $\triangleleft$

Now **DWCI** is not particularly interesting. Some of its extensions are. *One* of the usual extensions still works with this semantics, and this will give us a natural semantics for **C** which is **DWCI** + **B**

THEOREM 6.7 *The prefixing condition is sound and complete with respect to the axiom of prefixing, in the natural four-valued semantics.*

Proof: Completeness is easy — it involves showing that R in the almost canonical structure satisfies the prefixing condition under the assumption of prefixing. The proof in the previous chapter can be used for this, and the reader is referred there.

Soundness is an order of magnitude more difficult, and so we will work the details. Suppose that $R^2a(bc)d \Rightarrow R^2(ab)cd$ and that

$$g \not\models (A \to B) \to ((C \to A) \to (C \to B))$$

Then using our standard tree construction we can see that this is not satisfiable.

1 $g \not\models (A \to B) \to ((C \to A) \to (C \to B))$

2 $w \models A \to B$ [1]			4 $w \dashv (C \to A) \to (C \to B)$ [1]
3 $w \not\models (C \to A) \to (C \to B)$ [1]			5 $w \not\dashv A \to B$ [1]
6 $Rwxy$ [3]			34 $Rxyw$ [4]
			35 $x \models C \to A$ [4]
7 $x \models C \to A$ [3]		9 $x \dashv C \to B$ [3]	36 $y \dashv C \to B$ [4]
8 $y \not\models C \to B$ [3]		10 $y \not\dashv C \to A$ [3]	37 $Ruvy$ [36]
11 $Ruyv$ [8]		25 $Ruvx$ [9]	38 $u \models C$ [36]
		26 $u \models C$ [9]	39 $v \dashv B$ [36]
12 $u \models C$ [8]	14 $u \dashv B$ [8]	27 $v \dashv B$ [9]	40 $Rxut$ [34,
13 $v \not\models B$ [8]	15 $v \not\dashv C$ [8]	28 $Rvux$ [25,CI]	41 $Rtvw$ 37,B]
16 $Rxwy$ [6,CI]	21 $Ruwt$ [6,	29 $Rwvt$ [6,	42 $Ruxt$ [41,CI]
17 $Ruxt$ [6,	22 $Rtxv$ 11,B]	30 $Ruty$ 28,B]	43 $t \models A$ [35,38,42]
18 $Rtwv$ 16,B]	23 $t \dashv A$ [2,14,22]	31 $Rvwt$ [30,CI]	44 $v \not\dashv B$ [4,15,43]
19 $t \models A$ [7,12,18]	24 $v \dashv C$ [7,22,23]	32 $t \dashv A$ [2,27,31]	$\times$ [39,44]
20 $v \models B$ [2,18,19]	$\times$ [15,24]	33 $t \dashv A$ [10,30,32]	
$\times$ [13,20]		$\times$ [32,33]	

◁

CK and **R** are interestingly dual systems (see John Slaney's 'Finite Models for Non-Classical Logics' [150] to see *some* examples of the duality between them). The natural semantics gives another setting in which their duality can be exposed — for the expected way of restricting R to model these systems both fail, for 'dual' reasons.

To perform the extension to **CK**, we would need to expand the definition of a containment relation:

$$a \sqsubseteq b \Rightarrow \begin{cases} a \models p \;\Rightarrow\; b \models p & \text{for every propositional variable } p, \\ a \dashv b \;\Rightarrow\; b \models p & \text{for every propositional variable } p, \\ Rcbd \Rightarrow Rcad & \text{if } a \neq g, \\ Rcbd \Rightarrow c \sqsubseteq d & \text{if } a = g. \\ Rcda \Rightarrow Rcdb & \end{cases}$$

and then show that it did what we wished of it. An easy induction on the length of formulae shows that if $\sqsubseteq$ is a containment relation on $\langle g, W, R, \models, \dashv \rangle$ then $a \sqsubseteq b$ gives $a \models A \Rightarrow b \models A)$ and $a \dashv A \Rightarrow b \dashv A$ for every formula A.

Then, to extend using **K**, note what happens when you attempt to show soundness using the standard condition. Assume that the condition $Rabc \Rightarrow b \sqsubseteq c$ holds in an interpretation. We wish to show that $g \models B \to (A \to B)$. To do that, we need to show that (among other things) if $w \models B$, then $w \models A \to B$, and to do that, we need to show that if $Rxwy$ and $x \dashv B$, we have $y \dashv A$. This does not seem to be ensured. We have that $Rwxy$ by the condition for assertion, and so, $x \sqsubseteq y$ gives $y \dashv B$. But $Rxwy$ gives $w \sqsubseteq y$ and so $y \models B$. So, if we are sure that y is *consistent* (that is, for no B is $y \models B$ and $y \dashv B$), we can ensure that our condition holds (if vacuously). What we would need to do is show that in the presence of **K**, a consistency assumption on worlds could be made. However, once this is done, more than **CK** is captured. So it seems that **CK** escapes this modelling.

Dual problems beset the extension of **C** to its contraction-added (and more famous) cousin **R**. The obvious candidate to add to **C** to get **R** is **W**. In the context of **DWCI**, **W** is equivalent to $(A \circ A \to B) \to (A \to B)$ which in turn is equivalent to $A \to A \circ A$. Now the truth and falsity conditions for fusion can be deduced from those for implication. We wish to show that $A \to A \circ A$ hold at g, under the condition that $Rabc \Rightarrow Ra(ab)c$ (Which gives, when $b = g$, that $Raaa$.)

To get the conditional to hold, we need that for all w, if $w \dashv A \circ A$, then $w \dashv A$. Now $w \dashv A \circ A$ iff $w \models A \to {\sim}A$, which is simply that for all x, y where $Rxwy$, if $x \models A$, $y \dashv A$. In the context of **W**, all we have to go on is that $Rwww$, so we have that if $w \models A$, $w \dashv A$. This is *not* enough to show that $w \dashv A$ per say, for w may neither support nor undermine A. The soundness proof grinds to a halt at this point. Of course, if the *completeness* of each world is ensured, the soundness proof goes through, but again, more than **R** is captured if this line is followed.

So, the natural four-valued semantics is incredibly discerning — it will only brook a modification to give **C**, and none of the other standard deviant systems (like **R**) can be modelled in this way. Clearly, other possibilities ought to be examined.

6.4 Another Possibility?

As we promised in Chapter 2, the semantics gives rise to alternate modellings of negation. Here, the four-valued semantics shows that the negation given by * and the proof theoretical rules of Chapter 2 is not the only possible negation for our logics. Another possibility for the negation conditions for a conditional comes from considering the conditions for the *truth* of a conditional. If for $A \to B$ to be true at w we need for each x, y where $Rxwy$, if A is true in x then B is true in y, and if B is false in x then A is false in y, then for $A \to B$ to be false at w we should have a counterexample to this. Provided that we construe a counterexample as a situation in which the antecedent is true and the consequent false, then this is indeed a counterexample. A natural way of formulating this is as follows.

$$w \dashv A \to B \text{ iff for some } x, y \text{ where } Rxwyx \models A \text{ and } y \dashv B$$

The semantics with this condition gives more than **DW**. One thing we get is the rule

$$A \wedge {\sim}B \vdash {\sim}(A \to B).$$

Another extra is the axiom

$${\sim}(A \to C) \wedge (A \to {\sim}B) \to {\sim}(A \to B)$$

which holds in all of the model structures. This is a theorem of classical logic, but it is not a theorem of **R** or of **CK**,[2] and so, it is not a theorem of any systems weaker than these. It follows that again, systems like **R** and **CK** cannot be modelled along these lines — furthermore, nothing weaker than **R** or **CK** can be modelled with this semantics. Whatever they capture, the systems will be a new family, outside the standard relevant systems.

If we add $\perp$ to the language, our axiom can be recast as

$$\sim -A \wedge (A \to \sim B) \to \sim(A \to B)$$

or dually, as

$$(A \to B) \to -A \vee \sim(A \to \sim B)$$

This has a certain plausibility about it. If you read '$-A$' as "A is absurd" (which makes sense on the definition) then the axiom reads "If A is not absurd, and if A implies $\sim B$, then it is not the case that if A then B" or dually, "If (if A then B), then either A is absurd, or it is not the case that if A then $\sim B$."

The new axiom and rule *seem* to characterise the new conditions in the semantics, but completeness has not been proved. We will sketch the difficulty in proving it, so others can have a crack at it. Taking the almost canonical model structure as before, we must ensure that if $R\Sigma\Gamma\Delta$, then $C \in \Sigma$ and $\sim D \in \Delta$ ensure that $\sim(C \to D) \in \Gamma$. This does not seem to follow from our original definition of R, so we must add it as another restriction on R. This is no problem in itself. The difficulty arises with the proofs of the priming lemmas. We must have that if $A \to B \notin \Gamma$, then there are Σ and Δ where $A \in \Sigma$ and $B \notin \Delta$ *and* $R\Sigma\Gamma\Delta$. Proving that the modified R relation holds between the theories constructed with the old method seems impossible. The usual completeness proof does not work, and as none have yet been found, completeness for the new semantics must be left as an open problem.

6.5 Notes

[1]The connection between the two- and four-valued systems has a long history, as can be seen in section 3.2 of *Relevant Logics and their Rivals* [131]. I make no claim to originality for *that* in this section. Instead, I take the work in new directions by sketching out other possibilities for future research.

[2]It is simple to check its failure in the matrices **RM3** and $Ł_3$. (**RM3** is given by the t, b and f fragment of **BN4**, and it is a well known model of **R**.)

Chapter 7
Decidability

Time was when contributors to philosophical logic
needed to do no more
than demonstrate the appropriateness of a logic
for the abstract description of validity
across a chosen range.
In those days it sufficed
that language suitably regimented
had indeed the structure of the presented system.
Now, though, correctness is not enough.
In the terms of the new electronic pragmatism,
the truth may not be pure
but it had better be simple.
For "simple" read "computationally tractable."
— JOHN SLANEY "Vagueness Revisited" [146]

Decidability is often an interesting issue for non-classical logics. Once we distinguish between intensional and extensional bunching, checking sequents for provability becomes complex. Whether or not $X; Y \Vdash A$ is a different question to whether or not $X, Y \Vdash A$. This leads to some interesting problems for logics with contraction, but without other intensional rules (like weakening). In these logics, to check whether $X \Vdash A$, you have to check whether $X; X \Vdash A$ or $X; (X; X) \Vdash A$ or $X; (X; (X; X)) \Vdash A$ and so on. Because each of these sequents gives the original sequent by applications of contraction, but in the absence of weakening (or mingle) they are not necessarily *equivalent* as sequents. In logics like **R**, X and $X; X$ differ in force for many values of X. If your method of proof-search involves backtracking from a sequent to consider possible proofs of that sequent, this endless chain of non-equivalent sequents can become a problem. In certain contraction-added logics like **R**, this problem is serious. Strong validity in **R** is undecidable.[1]

In the absence of **W** this phenomenon does not worry us. In sequent-checking there is no need to go back in the search space checking infinite collections of sequents. Strong validity from each of our favourite logics is decidable. This result for **DW**, **TW**, **C** and **CK** was demonstrated by Ross Brady in 1991 [18, 19], and we will use his method in this chapter. We will also expand his method to show that strong validity from **EW**, $\mathbf{L}^+$ and $\mathbf{LI}^+$ are all decidable. This is a new result.

However, decidability vanishes once we move to weak validity. Urquhart has shown that weak validity in all logics between $\mathbf{TW}^+$ and **KR** is undecidable [7, 164]. (**KR** extends **R** by adding $A \wedge {\sim} A \to B$ as an axiom.) In weak validity there is *no* restriction on the number of uses of premises, so decidability fails in these contexts. It is still unknown whether or not weak validity in **DW** and **CK** is decidable.

We will examine Gentzen-style sequent calculi for each of our favourite logics. These calculi will give us a handle on proof search and yield the available decidability results.

7.1 The Positive Gentzenisation

The Gentzen system is reasonably tricky, as negation is difficult to model. For ease of presentation, we will define the positive system for our logics without negation (or quantification), and prove the decidability results for these systems. Then the tinkering to add negation will be relatively straightforward. This also makes sense, as we wish to prove decidability results for positive-only systems like $\mathbf{L}^+$ and $\mathbf{LI}^+$.

As always, the work hinges on proving cut elimination, and we will follow the proof procedure used originally by Dunn [4, 28] and Minc [171], and then by Meyer [85], Brady [18, 19] and Slaney [144, 148].

Let $\mathbf{X}^+$ be the logic we have in mind. This can by any of our favourite logics, without negation or quantification. The Gentzen system $\mathbf{GX}^+$ uses a number of *structural* rules, just like $\mathbf{X}^+$. Firstly, for all logics we have

$$\frac{X(Y,Y) \Vdash A}{X(Y) \Vdash A}\ \text{eW} \qquad \frac{X(W,(Y,Z)) \Vdash A}{X((W,Y),Z) \Vdash A}\ \text{eB}$$

$$\frac{X(Y,Z) \Vdash A}{X(Z,Y) \Vdash A}\ \text{eC} \qquad \frac{X(Y) \Vdash A}{X(Y,Z) \Vdash A}\ \text{eK}$$

$$\frac{X(Y) \Vdash A}{X(Y;0) \Vdash A}\ 0\text{I} \qquad \frac{X(Y;0) \Vdash A}{X(Y) \Vdash A}\ 0\text{E}$$

Then there are the intensional structural rules:

$$\frac{X(0;Y) \Vdash A}{X(Y;0) \Vdash A}\ 0\text{-swap right} \qquad \frac{X(Y;0) \Vdash A}{X(0;Y) \Vdash A}\ 0\text{-swap left}$$

$$\frac{X((W;Y);Z) \Vdash A}{X(W;(Y;Z)) \Vdash A}\ \mathbf{B} \qquad \frac{X(W;(Y;Z)) \Vdash A}{X((W;Y);Z) \Vdash A}\ \mathbf{B}^{dual}$$

$$\frac{X((W;Z);Y) \Vdash A}{X(W;(Y;Z)) \Vdash A}\ \mathbf{B}' \qquad \frac{X(W;(Y;Z)) \Vdash A}{X((W;Z);Y) \Vdash A}\ \mathbf{B}'^{dual}$$

$$\frac{X(W;(Y;Z)) \Vdash A}{X(Y;(W;Z)) \Vdash A}\ \mathbf{C} \qquad \frac{X(Y) \Vdash A}{X(Z;Y) \Vdash A}\ \mathbf{K}$$

We can add as many or as few of the structural rules as we like to get the usual range of logical systems.

As usual there is a single axiom scheme.

$$A \Vdash A \quad \text{Ax}$$

The logical rules are all the ones you would expect, given the difference between intensional and extensional bunching, and the move to a Gentzen calculus, which uses introduction rules on the left and right, instead of introduction and elimination rules on the right.

$$\frac{A; X \Vdash B}{X \Vdash A \to B}\ \Vdash\to \qquad \frac{X \Vdash A \quad Y(B) \Vdash C}{Y(X; A \to B) \Vdash C}\ \to\Vdash$$

$$\frac{X; A \Vdash B}{X \Vdash B \leftarrow A}\ \Vdash\leftarrow \qquad \frac{X \Vdash A \quad Y(B) \Vdash C}{Y(B \leftarrow A; X) \Vdash C}\ \leftarrow\Vdash$$

$$\frac{X \Vdash A \quad Y \Vdash B}{X; Y \Vdash A \circ B}\ \Vdash\circ \qquad \frac{X(A; B) \Vdash C}{X(A \circ B) \Vdash C}\ \circ\Vdash$$

$$\frac{X \Vdash A \quad Y \Vdash B}{X, Y \Vdash A \wedge B}\ \Vdash\wedge \qquad \frac{X(A, B) \Vdash C}{X(A \wedge B) \Vdash C}\ \wedge\Vdash$$

$$\frac{X \Vdash A}{X \Vdash A \vee B} \qquad \frac{X \Vdash B}{X \Vdash A \vee B}\ \Vdash\vee \qquad \frac{X(A) \Vdash C \quad X(B) \Vdash C}{X(A \vee B) \Vdash C}\ \vee\Vdash$$

$$\frac{X(0) \Vdash A}{X(t) \Vdash A}\ t\Vdash \qquad \frac{}{0 \Vdash t}\ \Vdash t$$

Call the Gentzen system $\mathbf{GX}^+$ (for $\mathbf{X}$ any of our usual logics). We'll say that a formula A is provable in $\mathbf{GX}^+$ just when $t \Vdash A$. We write this as $\vdash A$, as before. For example, $\vdash (A \to C) \wedge (B \to C) \to (A \vee B \to C)$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \Vdash A \quad C \Vdash C}{A; A \to C \Vdash C}}{A; (A \to C, B \to C) \Vdash C}}{A; (A \to C) \wedge (B \to C) \Vdash C} \quad \dfrac{\dfrac{\dfrac{B \Vdash B \quad C \Vdash C}{B; B \to C \Vdash C}}{B; (A \to C, B \to C) \Vdash C}}{B; (A \to C) \wedge (B \to C) \Vdash C}}{A \vee B; (A \to C) \wedge (B \to C) \Vdash C}}{(A \to C) \wedge (B \to C) \Vdash A \vee B \to C}}{(A \to C) \wedge (B \to C); 0 \Vdash A \vee B \to C}}{(A \to C) \wedge (B \to C); t \Vdash A \vee B \to C}}{t \Vdash (A \to C) \wedge (B \to C) \to (A \vee B \to C)}$$

We use 0-swap right and left to prove $(t \to A) \to A$ and $A \to (t \to A)$ respectively.

$$\dfrac{\dfrac{\dfrac{\dfrac{0 \Vdash t \quad A \Vdash A}{0; t \to A \Vdash A}}{t \to A; 0 \Vdash A}}{t \to A; t \Vdash A}}{t \Vdash (t \to A) \to A} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \Vdash A}{A; 0 \Vdash A}}{0; A \Vdash A}}{t; A \Vdash A}}{A \Vdash t \to A}}{A; 0 \Vdash t \to A}}{A; t \Vdash t \to A}}{t \Vdash A \to (t \to A)}$$

Finally, as another example of an axiom that uses an additional structural rule, we prove that $\vdash (A \to (B \to C)) \to (B \to (A \to C))$, using **C**.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \Vdash A \qquad \dfrac{B \Vdash B \quad C \Vdash C}{B; B \to C \Vdash C}}{B; (A; A \to (B \to C)) \Vdash C}}{A; (B; A \to (B \to C)) \Vdash C}}{B; A \to (B \to C) \Vdash A \to C}}{A \to (B \to C) \Vdash B \to (A \to C)}}{A \to (B \to C); 0 \Vdash B \to (A \to C)}}{A \to (B \to C); t \Vdash B \to (A \to C)}}{t \Vdash (A \to (B \to C)) \to (B \to (A \to C))}
$$

In order to prove that $\mathbf{GX}^+$ can derive any theorem of $\mathbf{X}^+$ it is helpful to use the cut rule:

$$
\dfrac{t \Vdash A \quad A \Vdash B}{t \Vdash B}\ \text{Cut}
$$

Then it is easy to prove the following theorem.

LEMMA 7.1 *If* $X \Vdash A$ *in* $\mathbf{X}^+$, *then* $X \Vdash A$ *in* $\mathbf{GX}^+ + \text{Cut}$.

Proof: The details of proof are left as an exercise for the interested reader. One way to do it is to show that the axioms of the Hilbert system are all provable (and we have just done some of this) and that the rules preserve provability. You need cut for *modus ponens.* ◁

The next thing to do is to show that if something is provable in the Gentzen system, we could do it in our other presentations of $\mathbf{X}^+$.

LEMMA 7.2 *If* $X \Vdash A$ *in* $\mathbf{GX}^+$, *then* $X \Vdash A$ *in* $\mathbf{X}^+$.

Proof: This involves showing that the Gentzen rules are weakly valid. It is simple, but tedious. The methods used for the interpretation of the natural deduction systems in the Hilbert systems suffices. ◁

7.2 Cut Elimination, Part 1

The significant result involves showing that if $X \Vdash A$ in $\mathbf{GX}^+ + \text{Cut}$, then $X \Vdash A$ in $\mathbf{GX}^+$ alone. To prove this, we use a more liberal rule, called 'mix,' and we show that if $X \Vdash A$ in $\mathbf{GX}^+ + \text{Mix}$, then $X \Vdash A$ in $\mathbf{GX}^+$. This will then give us the equivalence between $\mathbf{GX}^+$, and our other presentations of $\mathbf{X}^+$. To explain the mix rule, we need a definition.

Definition 7.1 If X is a bunch, and $\mathcal{A}$ is a (possibly empty) set of occurrences of the formula A in X, then $X(Y|\mathcal{A})$ is the bunch given by replacing those occurrences of A by the bunch Y. (To give a rigourous definition of this, we need to define 'occurrence.' Anderson, Belnap and Dunn palm this off to Quine [4], but I recommend Wetzel's more recent definition as a formalisation of the concept [165].)

The mix rule is then:

$$\frac{X \Vdash A \quad Y \Vdash B}{Y(X|\mathcal{A}) \Vdash B}\ \text{Mix}$$

where $\mathcal{A}$ is a set of of occurrences of A in Y. Every instance of cut is an instance of mix.

In order to show that $\mathbf{GX}^+$ + Mix gives us no more than $\mathbf{GX}^+$, we need to show that proofs with mixes in them can be rewritten without them. To do this, we will show that a proof with a mix in it can always have the mix rule either 'pushed back' to earlier in the proof, or done away with altogether. To aid us in our proof, we will show that the rules in $\mathbf{GX}^+$ allow us to push mixes past them. This requires a rather detailed analysis of what inferences and rules really are. (This analysis is lifted from *Entailment* volume 1 [4].)

Definition 7.2

- An *inference* is an ordered pair consisting of a set of sequents (the *premises*) and another sequent (the *conclusion*).
- A *rule* is a set of inferences. An inference is an *instance* of any rule of which it's a member.
- Let Inf be an inference which is an instance of the rule Ru. A *conclusion parameter* is a formula occurrence in the conclusion of Inf which is not the newly introduced "principal formula" of one of the rules, nor in the bunch Z that is introduced by eK or **K** (if present). A *premise parameter* is a formula occurrence in a premise of Inf which "matches" (spend a leisurely afternoon rigorously defining this notion) a conclusion parameter in Inf.

An example is illustrative. Take the inference

$$\frac{D, B \Vdash A \qquad D; (B, C) \Vdash C}{D; (((D, B); A \to B), C) \Vdash C}$$

It is an instance of $(\to \Vdash)$. The conclusion parameters are all the formula occurrences in the conclusion apart from $A \to B$. The first D in the conclusion matches the D in the *second* premise, the bunch D, B in the conclusion matches the D, B in the antecedent of the first premise, the C in the antecedent of the conclusion matches the C in the antecedent of the second premise, and the C in the consequent of the conclusion matches the C in the consequent of the second premise. There are no other matches.

To eliminate mixes we need to 'push' them above occurrences of other rules. For this purpose it is helpful to know when formulae can be replaced by bunches in rules. To explain when this is possible, we need some more definitions.

Definition 7.3 Let Inf be an inference. For example,

$$\frac{X_1 \Vdash A_1 \ \cdots \ X_n \Vdash A_n}{X_{n+1} \Vdash A_{n+1}}$$

and let $\mathcal{B}$ be a set of conclusion parameters, all of which occur in X_{n+1}. For each i, let $\mathcal{B}_i$ be the set of premise or conclusion parameters in X_i matching at least one of those in $\mathcal{B}$. For any bunch Y, define $Inf(Y|\mathcal{B})$ as

$$\frac{X_1(Y|\mathcal{B}_1) \Vdash A_1 \quad \cdots \quad X_n(Y|\mathcal{B}_n) \Vdash A_n}{X_{n+1}(Y|\mathcal{B}_{n+1}) \Vdash A_{n+1}}$$

So, $Inf(Y|\mathcal{B})$ is given by replacing each of the parameters matching those in $\mathcal{B}$ by Y. The rule Ru is said to be *left regular* if for every inference Inf and every set $\mathcal{B}$ of conclusion parameters occurring in the antecedent of Inf, if Inf is an instance of Ru, so is $Inf(Y|\mathcal{B})$.

To define "right regularity," we need a little more subtlety. Let Inf be as given above. Let C be a formula, and Y a bunch. Let $\mathcal{B}$ be the unit set of some formula occurrence in Y. If A_i matches the conclusion parameter A_{n+1} then S_i is $Y(X_i|B) \Vdash C$. If not, S_i is $X \Vdash A_i$. (Note that A_{n+1} is taken to match itself, so S_{n+1} is $Y(X_{n+1}|B) \Vdash C$.) Then $Inf(C, Y, \mathcal{B})$ is defined as

$$\frac{S_1 \quad \cdots \quad S_n}{S_{n+1}}$$

So, $Inf(C, Y, \mathcal{B})$ is given by systematically replacing parameters matching those in the consequent of the conclusion by C, and simultaneously embedding the antecedents of these sequents in a larger context, represented by the occurrence of a formula occurrence in Y.

A rule Ru is *right regular* if whenever it has Inf as an instance, it has $Inf(C, Y, \mathcal{B})$ as an instance, for all C, Y and $\mathcal{B}$ a unit set of a formula occurrence in Y.

Now given that this is out of the way, we can prove the following lemma:

LEMMA 7.3 *Every rule of* $\mathbf{GX}^+$ *is both left and right regular.*

Proof: The proof is by inspection. Each of the 0-rules are left regular because we do not allow 0 to be replaced by a bunch. In this way we can encode the special properties of t (like those given by 0-swap rules) without worrying whether we can replace t by a bunch or not.

As one part of the proof we show that $(\rightarrow\Vdash)$ is both left and right regular. Take an instance

$$\frac{X \Vdash A \quad Y(B) \Vdash C}{Y(X; A \rightarrow B) \Vdash C}$$

For left regularity note that the conclusion parameters in the antecedent must occur in $Y(-)$ and X. Let the substitution of bunches for conclusion parameters transform these to $Y^*(-)$ and X^*. Then it is clear that

$$\frac{X^* \Vdash A \quad Y^*(B) \Vdash C}{Y^*(X^*; A \rightarrow B) \Vdash C}$$

is also an instance of $(\rightarrow\Vdash)$. Similarly, for right regularity, the right premise $Y(B) \Vdash C$ has the matching consequent, so it is this and the conclusion that gets embedded in the larger context — say $Z(-)$. Then it is clear that

$$\frac{X \Vdash A \quad Z(Y(B)) \Vdash D}{Z(Y(X; A \rightarrow B)) \Vdash D}$$

is also an instance of $(\to\Vdash)$ as desired. Checking the other rules is merely tedious, so we assume it is done. ◁

To prove that mix can be eliminated from our proofs, we need to define give a rigorous definition of what it is to 'push' a mix backward in a proof. To do this, we need some kind of handle on how far down a mix comes in a proof. This means defining the concept of rank.

Definition 7.4 Let Der be a derivation, with S as its final sequent. Unless S is an axiom, take Inf to be the inference of which S is the conclusion, and take $\mathcal{A}$ to be a set of formula occurrences in S. We define the *rank of* $\mathcal{A}$ *in* Der as follows:

- If $\mathcal{A}$ is empty, its rank is 0.
- If $\mathcal{A}$ is non-empty, and contains only formula occurrences that are introduced by eK, **K** (if present) or a logical rule, or if S is an axiom, the rank of $\mathcal{A}$ is 1.
- Otherwise, let $S_1, \ldots, S_n$ be the premises of Inf, for each i let Der_i be the subproof terminating S_i, and let $\mathcal{A}_i$ be the (possibly empty) set of premise parameters in S_i that match at least one member of $\mathcal{A}$ (at least one $\mathcal{A}_i$ will be non-empty, granted that we are in this case). Let r_i be the rank of $\mathcal{A}_i$ in Der_i. Take the rank of $\mathcal{A}$ in Der to be $1 + \max_{1\leq i\leq n} r_i$.

In words, the rank of a set of formula occurrences in the definition is the length of the longest path upward in the derivation of matching formulae in that set, plus one. (And it's zero if the set is empty).

This gives us enough to prove our theorem.

Theorem 7.4 *If* $X \Vdash A$ *in* $\mathbf{GX}^+ + \text{Mix}$ *then* $X \Vdash A$ *in* $\mathbf{GX}^+$.

Proof: It suffices to show that if a sequent can be proved in $\mathbf{GX}^+ + \text{Mix}$ with a proof ending in a mix, then this sequent can be proved without that mix. Take the final mix and label the premises and the conclusion as follows:

$$\frac{[L]\quad X \Vdash A \qquad [R]\quad Y \Vdash B}{[C]\quad Y(X|\mathcal{A}) \Vdash B}\ \text{Mix}$$

Let k be the rank of the consequent occurrence of A in the subproof ending in L, and let j be the rank of the occurrences $\mathcal{A}$ in the subproof ending with R. The rank of the mix is $k + j$.

In the proof, we will use a double induction. The outer induction hypothesis is that all mixes of formulae less complex than A are eliminable. The inner induction is that given A, all mixes of rank less than the mix we are considering are eliminable.

Given the mix displayed above, these cases exhaust the options:

Case 1. Either $k = 1$ and L is an axiom, or $j = 0$ or $j = 1$, with R an axiom. In this case, either $C = R$ or $C = L$, so the mix is unnecessary. So, it can be eliminated.

Case 2a. R is given by eK, and $j = 1$. It follows that all occurrences of A in $\mathcal{A}$ have been weakened in, and so, we can derive C from R by eK. Again, the mix is unnecessary.

Case 2b. R is given by $\mathbf{K}$, and $j = 1$. It follows that all occurrences of A in $\mathcal{A}$ have been weakened in, and so, we can derive C from R by $\mathbf{K}$. Again, the mix is unnecessary.

Case 3. $j \geq 2$. Suppose the derivation to R ends in the inference Inf, an instance of Ru

$$\frac{Y_1 \Vdash B_1 \;\cdots\; Y_n \Vdash B_n}{Y \Vdash B}$$

Let $\mathcal{A}'$ be the set of occurrences of A that are conclusion parameters in Inf. So $\mathcal{A} \backslash \mathcal{A}'$ are introduced by Inf. Every premise of $\mathit{Inf}(X|\mathcal{A}')$ is provable by a mix with L as the minor premise (the rank of the these mixes will be less than $k + j$, so they are eliminable), or without a mix, if $\mathcal{A}'$ has no occurrences in that premise. As Ru is left regular, $\mathit{Inf}(X|\mathcal{A}')$ is also an instance of Ru, so $Y(X|\mathcal{A}') \Vdash B$ is provable without a mix. If $\mathcal{A} \backslash \mathcal{A}' = \varnothing$, we are done. If not, let $\mathcal{A}_0$ be the occurrences of A in $Y(X|\mathcal{A}')$ that correspond to those in $\mathcal{A}_0$. These are introduced by $\mathit{Inf}(X|\mathcal{A}')$, so the rank of $\mathcal{A}_0$ in this proof is 1. As $k+1 < k+j$, this mix can also be eliminated.

Case 4. $k \geq 2$ and $j = 1$, with R not an axiom, and R not given by eK or $\mathbf{K}$. It follows that R is given by a logical rule (not a structural rule), and $\mathcal{A}$ is the unit set of the formula occurrence newly introduced into Y. Let Inf (an instance of Ru) be the rule that gives L. By the hypothesis applied to the premises of Inf containing a parameter matching the A displayed in L, we get the provability of every premise of $\mathit{Inf}(B, Y, \mathcal{A})$. But this is an instance of Ru by right regularity, so we get the provability of C, which is its conclusion.

Case 5. $k = 1$, $j = 1$, R is not an axiom, and it is not given by E-weakening or $\mathbf{K}$. So, R is given by a logical rule, and $\mathcal{A}$ is the unit set containing the formula occurrence newly introduced into Y, and the displayed A in L is *also* given by a logical rule. These logical rules must introduce the same connective.

5.1 $A = A_1 \circ A_2$ or $A = A_1 \wedge A_2$. Then L and R are given as follows:

$$\frac{X_1 \Vdash A_1 \quad X_2 \Vdash A_2}{X_1 * X_2 \Vdash A_1 \star A_2} \qquad \frac{Y(A_1 * A_2) \Vdash B}{Y(A_1 \star A_2) \Vdash B}$$

Where $\star$ is one of $\circ$ or $\wedge$ and $*$ is either ; or , as appropriate. The hypothesis gives the provability of $Y(X_1 * A_2) \Vdash B$ (eliminating a mix with premises $X_1 \Vdash A_1$ and $Y(A_1 * A_2) \Vdash B$) and then the provability of $Y(X_1 * X_2) \Vdash B$ (eliminating the mix with premises $X_2 \Vdash A_2$ and $Y(X_1 * A_2) \Vdash B$).

5.2 $A = A_1 \vee A_2$. In this case the proofs inferences giving L and R are (without loss of generality):

$$\frac{X \Vdash A_1}{X \Vdash A_1 \vee A_2} \qquad \frac{Y(A_1) \Vdash B \quad Y(A_2) \Vdash B}{Y(A_1 \vee A_2) \Vdash B}$$

Then the hypothesis shows us that the mix that delivers $Y(X) \Vdash B$ from $X \Vdash A_1$ and $Y(A_1) \Vdash B$ can be eliminated, so we replace the old mix with that proof.

5.3 $A = A_1 \to A_2$. (The case for $A = A_1 \leftarrow A_2$ is similar, and is left to the reader.) The inferences to L and R are:

$$\frac{A_1; X \Vdash A_2}{X \Vdash A_1 \to A_2} \qquad \frac{Z \Vdash A_1 \quad Y(A_2) \Vdash B}{Y(Z; A_1 \to A_2) \Vdash B}$$

By hypothesis, the proof of $Z; X \Vdash A_2$ (which is mixed from $A_1; X \Vdash A_2$ and $Z \Vdash A_1$) is eliminable, and we can then eliminate the mix that uses this and $Y(A_2) \Vdash B$ to give $Y(Z; X) \Vdash B$, as we desired.

5.4 $A = t$. Then the left sequent is $0 \Vdash t$, and the right is $Y(t) \Vdash B$, which was deduced from $Y(0) \Vdash B$. The conclusion of the mix is $Y(0) \Vdash B$, which we have already shown. So, the mix is eliminated.

So, each possible mix can be eliminated. So, our theorem is proved. ◁

7.3 Decidability, Part 1

The Cut Elimination Theorem is a Good Thing, because it enables us to prove lots more. For example, we can show that the positive parts of our proof theory conservatively extend the parts containing only the intensional connectives. In addition, we get the decidability result. For this, we need to understand the structure of proofs, and this means introducing a number of concepts.

Definition 7.5

- A sequent $X \Vdash A$ is an *i-only-sequent* if it is free of e-bunches and the extensional connectives $\wedge$ and $\vee$.
- The *e-equivalents* of a bunch (or sequent) are those bunches (or sequents) to which it can be converted by means using the rules eB and eC.
- A bunch can be *eW-reduced* if it contains a sub-bunch of the form X, X (at some depth). It *is eW-reduced* if it cannot be eW-reduced any further.
- An e-bunch is *typically boring* if it is of the form $(X, (X, (X, \ldots, X)))$, where X is not an e-bunch. An e-bunch is *boring* if it is e-equivalent to a typically boring e-bunch. An e-bunch is *interesting* if it not boring.
- A proof of an i-only-sequent is *confused* if it contains an e-bunch as a sub-bunch at some depth.
- A proof of an i-only-sequent is *insane* if it contains an interesting e-bunch as a sub-bunch at some depth.
- A tree of sequents has the *subformula property* if for every sequent in the tree, the formulae in sequents above it are all subformulae of the formulae in that sequent.

LEMMA 7.5 *All proofs in* $\mathbf{GX}^+$ *have the subformula property.*

Proof: A simple inspection of the rules shows that the formulae in each premise are subformulae of formulae in the conclusion. By the transitivity of subformulahood, the proof is done. ◁

It is this that makes Gentzen systems valuable; it gives us a measure of control over proofs. For example, we can prove the following.

LEMMA 7.6 *No proof of an i-only-sequent is insane.*

Proof: Firstly, by the subformula property, the proof will not make use of the rules for extensional connectives.

The conclusion of the proof is free of e-bunches, so it cannot contain an e-bunch. Suppose that the conclusion of an inference in the proof contains no interesting e-bunches. The only rules in this proof that change e-bunches are the e-structural rules, and inspecting them shows that the premises of these cannot contain interesting e-bunches if their conclusions contain only boring e-bunches. So, there is no way that interesting e-bunches can appear in a proof of an i-only-sequent. ◁

LEMMA 7.7 *If an i-only-sequent has a confused proof, it also has a proof that is not confused.*

Proof: The proof cannot be insane (by the previous lemma) so the only e-bunches it contains are boring. Every boring e-bunch X is e-equivalent to some $(Y, (Y, (Y, \ldots, Y)))$ where Y is not an e-bunch. Transform the proof as follows: replace each such X in the proof by Y. The result is a proof that is not confused. (In the old proof e-bunches must be introduced by eK. The corresponding step in the new proof has the same conclusion as its premise. In the old proof, e-bunches are removed by eW, the corresponding step in the new proof is valid; again its premise is its conclusion. The other rules that change e-bunches in the old proof simply have no effect in the new proof, and all other steps in the new proof are instances of the same rules that are used in the old proof.) This proof has the same conclusion as the old one, so this is the proof we are looking for. ◁

So, we have a new conservative extension result.

COROLLARY *Each sequent in the intensional vocabulary is provable in a purely intensional vocabulary in* $\mathbf{GX}^+$. *So,* $\mathbf{GX}^+$ *conservatively extends* $\mathbf{GX}_{\rightarrow\circ\leftarrow t}$

There is another simple corollary of the conservative extension result. We have a different proof of our primeness result.

THEOREM 7.8 $\vdash A \lor B$ *if and only if* $\vdash A$ *or* $\vdash B$.

Proof: A simple inspection of the rules of $\mathbf{GX}^+$ suffices. Any proof that $t \Vdash A \lor B$ *must* use $\lor \Vdash$ to put $A \lor B$ in the consequent of the sequent. So, it must have come from either $t \Vdash A$, $t \Vdash B$, $0 \Vdash A$ or $0 \Vdash B$. ◁

To show whether or not $X \Vdash A$ we search for sequents that it could have been derived from. Most of the rules at most *increase* the amount of material in a sequent, so running them backwards does not increase the amount of material. These rules do not cause the search space to balloon out uncontrollably. The only offending exceptions are eW and 0E. To prove decidability we need to get some amount of control over the number of times these rules must be used in a proof.

Definition 7.6

- A bunch X is said to be *eW semi-reduced* if none of its e-equivalents contain sub-bunches of the form $Y, (Y, Y)$.
- A bunch X is said to be 0 *reduced* if it contains no sub-bunches of the form $Y; 0$.
- A bunch X is said to be 0 *semi-reduced* if it contains no sub-bunches of the form $(Y; 0); 0$.
- A bunch is said to be *reduced* (*semi-reduced*) if it is both eW and 0 reduced (or semi-reduced).
- A sequent is reduced or semi-reduced iff its antecedent is, and a proof is reduced or semi-reduced iff each of its sequents are.

Clearly, given a sequent S, there is a corresponding reduced sequent $r(S)$ (found by performing as many e-contractions and 0-eliminations on S and its e-equivalents as possible) that is provable iff S is provable.

Lemma 7.9 *Any provable reduced sequent has a semi-reduced proof.*

Proof: By induction on the construction of proofs. A quick inspection will show that for any instance of a rule, with premises S_1 and S_2 and conclusion S_3, there is another proof from $r(S_1)$ and $r(S_2)$ to $r(S_3)$ (or $r(S_1) = r(S_3)$) where the intermediate steps are all structural rules, with at most semi-reduced sequents. Given this, it is clear that any proof of a reduced sequent can be 'reduced' step by step to ensure that each step is at worst semi-reduced. ◁

This result gives us some kind of control over the search space of possible proofs of a reduced sequent. We now know that we only need search through semi-reduced proofs.

Definition 7.7 The *loose equivalents* of a semi-reduced sequent are those semi-reduced sequents from which the original sequent can be obtained by any of the structural rules available in the logic.

Lemma 7.10 *Every semi-reduced sequent has only a finite number of semi-reduced loose equivalents, and these can be effectively listed.*

Proof: By inspection of the structural rules. None of them adds new material apart from eW and 0E, and the endless application of eW or 0E (backwards) soon leads to the loss of semi-reduction. As sequents are finite, and structural rules give only different ways of arranging the finite material, there is only a finite number of semi-reduced sequents loosely equivalent to our original sequent. To list them, take the converse of each structural rule, apply it to the original sequent, and if the resulting sequent is semi-reduced, record that sequent. Continue this process with each listed sequent. After a finite number of stages, the possibilities will be exhausted. ◁

Note that in this proof, had **W** been present as a structural rule, this argument would not have gone through, because running **W** backwards indefinitely leads to an infinite number of distinct sequents. Adding **W** to our logics can lead to undecidability. Thankfully for us, in the absence of **W** decidability is simple to prove. We need just one more definition.

Definition 7.8 The *complexity* of a sequent S is the number of times connectives or t appears in its reduced bunch $r(S)$.

LEMMA 7.11 *If S is of complexity n and S' is equivalent to S, then S' is of complexity at most n.*

Proof: None of the structural rules listed decrease complexity. The extensional contraction rule eW may be thought to, but if S comes from S' by eW, then $r(S)$ and $r(S')$ contain the same number of logical constants, and so, S and S' have the same complexity. ◁

THEOREM 7.12 $\mathbf{X}^+$ *is decidable.*

Proof: By induction on the complexity of sequents. Clearly all sequents of complexity 0 are decidable. Now assume that all sequents of complexity less than n are decidable. Given that S is of complexity n, consider $r(S)$. If the complexity of $r(S)$ is less than n, then by hypothesis, whether or not it is provable is decidable, and we are done. If not, consider the finite number of loose equivalents of $r(S)$. S is provable if and only if at least one of these will be provable. Now each of these sequents S' are no more complex than S, only a finite number of logical rules could have been used to derive them. For each S' consider the possibilities. The premises of these rules all have complexity less than that of S', so they are decidable. It is then a matter of inspection to see whether or not any S' is derivable. Hence, we have a decision procedure for S. ◁

7.4 The Whole Gentzenization

The last set of results were nice, but not enough. To model our logics, we need to have a handle on negation. Thankfully, this is tedious, but not difficult, given the material we have covered. The difficulty comes from not being able to use the traditional rules for negation because these would yield only an intuitionistic negation, in the absence of bunches on the right. And we are stuck with asymmetric systems, because there seems to be no way to state the mix rule in a right-bunched system in a way that is both valid and eliminable [18, 19]. So, we need another way to model negation. Ross Brady has given us one way to do this. We need to extend the system to involved *signed* formulae instead of just formulae. Instead of treating negation like any other connective, we can 'wire it in' to the statements of the other rules by signing formulae with T or F to denote truth and falsity, to ensure that it has the desired properties (such as the de Morgan identities, contraposition and double negation). So, we end up with these rules:

$$\frac{X \Vdash TA \quad Y \Vdash TB}{X, Y \Vdash T(A \wedge B)} \Vdash T\wedge \qquad \frac{X(TA, TB) \Vdash SC}{X(TA \wedge TB) \Vdash SC} T\wedge \Vdash$$

$$\frac{X(FA) \Vdash SC \quad X(FB) \Vdash SC}{X(F(A \wedge B)) \Vdash SC} F\wedge \Vdash \qquad \frac{X \Vdash FA}{X \Vdash F(A \wedge B)} \qquad \frac{X \Vdash FB}{X \Vdash F(A \wedge B)} \Vdash F\wedge$$

where SC denotes an arbitrary formula, C, with an arbitrary sign, S. We're nearly done, but there's another subtlety to deal with. The intensional connectives need more work.

For example, how can a false conditional arise in the consequent of a sequent? If this is our rule

$$\frac{X \Vdash TA \quad Y \Vdash FB}{X;Y \Vdash F(A \to B)}$$

we could show that $A' \to A, B' \to \sim B \vdash A' \circ B' \to \sim(A \to B)$, which is not valid in logics like **TW** or **EW**. To model negated conditionals, and fusion, we need to introduce *another* structural connective ':' which will mimic the behaviour of a negated conditional. In particular, $A{:}\,B$ will represent $\sim(A \to \sim B)$ (which we write now as $A \oplus B$) and so, we will be able to model negated conditionals and negated fusions in our Gentzen system. We revise the definitions of bunches to constitute structures of *signed* formulae and 0 bunched together using any of the three bunching operators, ',' (extensional) and ';' and ':' (both intensional). To give us contraposition for the conditional we admit another structural rule.

$$\frac{X(Y{:}\,Z) \Vdash A}{X(Z{:}\,Y) \Vdash A}\ jC$$

Then we expand our logical rules to deal with the positive and negative cases.

$$\frac{TA;X \Vdash TB \quad FB;X \Vdash FA}{X \Vdash T(A \to B)}\ \Vdash T\to \qquad \frac{X \Vdash TA \quad Y \Vdash FB}{X{:}\,Y \Vdash F(A \to B)}\ \Vdash F\to$$

$$\frac{X \Vdash TA \quad Y(TB) \Vdash SC}{Y(X;T(A \to B)) \Vdash SC} \qquad \frac{X \Vdash FB \quad Y(FA) \Vdash SC}{Y(X;T(A \to B)) \Vdash SC}\ T\!\to \Vdash$$

$$\frac{X(TA{:}\,FB) \Vdash SC}{X(F(A \to B)) \Vdash SC}\ F\!\to \Vdash \qquad \frac{X \Vdash TA \quad Y \Vdash TB}{X;Y \Vdash T(A \circ B)}\ \Vdash T\circ \qquad \frac{X{:}\,TA \Vdash FB}{X \Vdash F(A \circ B)}\ \Vdash F\circ$$

$$\frac{X(TA;TB) \Vdash SC}{X(T(A \circ B)) \Vdash SC}\ T\circ \Vdash \qquad \frac{Y(FB) \Vdash SC \quad X \Vdash TA}{Y(X{:}\,F(A \circ B)) \Vdash SC}\ F\circ \Vdash$$

$$\frac{X \Vdash TA \quad Y \Vdash TB}{X,Y \Vdash T(A \wedge B)}\ \Vdash T\wedge \qquad \frac{X(FA) \Vdash SC \quad X(FB) \Vdash SC}{X(F(A \wedge B)) \Vdash SC}\ F\wedge \Vdash$$

$$\frac{X(TA,TB) \Vdash SC}{X(T(A \wedge B)) \Vdash SC}\ T\wedge \Vdash \qquad \frac{X \Vdash FA}{X \Vdash F(A \wedge B)} \qquad \frac{X \Vdash FB}{X \Vdash F(A \wedge B)}\ \Vdash F\wedge$$

$$\frac{X \Vdash TA}{X \Vdash T(A \vee B)} \qquad \frac{X \Vdash TB}{X \Vdash T(A \vee B)}\ \Vdash T\vee \qquad \frac{X \Vdash FA \quad Y \Vdash FB}{X,Y \Vdash F(A \vee B)}\ \Vdash F\vee$$

$$\frac{X(TA) \Vdash C \quad X(TB) \Vdash SC}{X(T(A \vee B)) \Vdash SC}\ T\vee \Vdash \qquad \frac{X(FA,FB) \Vdash SC}{X(F(A \vee B)) \Vdash SC}\ F\vee \Vdash$$

$$\frac{X(FA) \Vdash SC}{X(T{\sim}A) \Vdash SC}\ T{\sim} \Vdash \qquad \frac{X(TA) \Vdash SC}{X(F{\sim}A) \Vdash SC}\ F{\sim} \Vdash \qquad \frac{X \Vdash FA}{X \Vdash T{\sim}A}\ \Vdash T{\sim}$$

$$\frac{X \Vdash TA}{X \Vdash F{\sim}A}\ \Vdash F{\sim} \qquad \frac{X(0) \Vdash SA}{X(Tt) \Vdash SA}\ Tt \Vdash \qquad \frac{}{0 \Vdash Tt}\ \Vdash Tt$$

Proofs in the system get to be rather complex rather quickly. The $(\Vdash T \rightarrow)$ rule has *two* premises (which are necessary for the proof of cut elimination, because a true conditional on the left could have come in two ways), and this makes proofs of nested conditionals rather long and involved.

To strengthen the logic to **TW** we need to add not just **B** and **B′** but also two new structural rules governing the interaction between the two kinds of intensional bunching:

$$\frac{X(Y\colon(Z;W)) \Vdash SA}{X((Y\colon Z);W) \Vdash SA}\ \mathbf{Bij} \qquad \frac{X((Z;Y)\colon W) \Vdash SA}{X(Y\colon(Z\colon W)) \Vdash SA}\ \mathbf{B'ij}$$

These rules are valid in **TW**. For the first, we need just show that $\vdash (A \oplus B) \circ C \rightarrow A \oplus (B \circ C)$. Here we argue as follows. Suffixing in **TW** gives $\vdash (B \rightarrow B \circ C) \rightarrow \big((B \circ C \rightarrow {\sim} A) \rightarrow (B \rightarrow {\sim} A)\big)$, and the fusion definition gives $\vdash C \rightarrow (B \rightarrow B \circ C)$. So transitivity gives $\vdash C \rightarrow \big((B \circ C \rightarrow {\sim} A) \rightarrow (B \rightarrow {\sim} A)\big)$. Contraposing three times leads to $\vdash C \rightarrow \big({\sim}(A \rightarrow {\sim} B) \rightarrow {\sim}(A \rightarrow {\sim}(B \circ C))\big)$, which when residuated and rewritten gives $\vdash (A \oplus B) \circ C \rightarrow A \oplus (B \circ C)$ as desired.

For the second, we need to show that $\vdash A \oplus (B \oplus C) \rightarrow (B \circ A) \oplus C$ or equivalently, that $\vdash \big(B \circ A \rightarrow {\sim} C\big) \rightarrow \big(A \rightarrow (B \rightarrow {\sim} C)\big)$. But this is easy:

1	(1)	$B \circ A \rightarrow {\sim} C$	A
2	(2)	A	A
3	(3)	B	A
3; 2	(4)	$B \circ A$	2, 3 ∘I
(3; 2); 1	(5)	${\sim} C$	1, 4 MP
3; (2; 1)	(6)	${\sim} C$	5 **B**
1	(7)	$A \rightarrow (B \rightarrow {\sim} C)$	2, 3, 6 CP twice

In the other direction, we utilise **B′ij** and both **B** and **B′** to prove prefixing. Because the proof is long, we will divide it into pieces. Here is the root:

$$\cfrac{\cfrac{\cfrac{F((C \rightarrow A) \rightarrow (C \rightarrow B)) \Vdash F(A \rightarrow B)}{0; F((C \rightarrow A) \rightarrow (C \rightarrow B)) \Vdash F(A \rightarrow B)}}{Tt; F((C \rightarrow A) \rightarrow (C \rightarrow B)) \Vdash F(A \rightarrow B)} \quad \cfrac{\cfrac{T(A \rightarrow B) \Vdash T((C \rightarrow A) \rightarrow (C \rightarrow B))}{0; T(A \rightarrow B) \Vdash T((C \rightarrow A) \rightarrow (C \rightarrow B))}}{Tt; T(A \rightarrow B) \Vdash T((C \rightarrow A) \rightarrow (C \rightarrow B))}}{Tt \Vdash T((A \rightarrow B) \rightarrow ((C \rightarrow A) \rightarrow (C \rightarrow B)))}$$

Then the left branch of the proof proceeds upward to axioms like this.

$$\cfrac{\cfrac{\cfrac{\cfrac{TC \Vdash TC \quad \cfrac{TA \Vdash TA \quad FB \Vdash FB}{TA\colon FB \Vdash F(A \rightarrow B)}}{(TC; T(C \rightarrow A))\colon FB \Vdash F(A \rightarrow B)}}{T(C \rightarrow A)\colon (TC\colon FB) \Vdash F(A \rightarrow B)}}{T(C \rightarrow A)\colon F(C \rightarrow B) \Vdash F(A \rightarrow B)}}{F((C \rightarrow A) \rightarrow (C \rightarrow B)) \Vdash F(A \rightarrow B)}$$

And the right branch goes up to

$$\frac{T(C \rightarrow A); T(A \rightarrow B) \Vdash T(C \rightarrow B) \quad F(C \rightarrow B); T(A \rightarrow B) \Vdash F(C \rightarrow A)}{T(A \rightarrow B) \Vdash T((C \rightarrow A) \rightarrow (C \rightarrow B))}$$

This, in turn, goes up on the left to

$$
\dfrac{
\dfrac{\dfrac{TC \Vdash TC \quad \dfrac{TA \Vdash TA \quad TB \Vdash TB}{TA; T(A \to B) \Vdash TB}}{(TC; T(C \to A)); T(A \to B) \Vdash TB}}{TC; (T(C \to A); T(A \to B)) \Vdash TB}
\quad
\dfrac{\dfrac{FB \Vdash FB \quad \dfrac{FA \Vdash FA \quad FC \Vdash FC}{FA; T(C \to A) \Vdash FC}}{(FB; T(A \to B)); T(C \to A) \Vdash FC}}{FB; (T(C \to A); T(A \to B)) \Vdash FC}
}{T(C \to A); T(A \to B) \Vdash T(C \to B)}
$$

and on the right to

$$
\dfrac{\dfrac{\dfrac{FB \Vdash FB \quad \dfrac{TC \Vdash TC \quad FA \Vdash FA}{TC{:}\, FA \Vdash F(C \to A)}}{TC{:}\, (FB; T(A \to B) \Vdash F(C \to A)}}{(TC{:}\, FB); T(A \to B) \Vdash F(C \to A)}}{F(C \to B); T(A \to B) \Vdash F(C \to A)}
$$

which completes the proof. As you can see, proofs in this Gentzen system can be rather long. However, the system has some pleasing properties which can help us prove important things about the logics with negation.

To go up to **EW**, in addition to adding 0-swap right, we need to add its colon analogue

$$
\frac{X(0{:}\, Y) \Vdash SA}{X(Y; 0) \Vdash SA} \; 0\mathbf{ij}
$$

With this structural rule we can then prove that $Tt \Vdash T((t \to A) \to A)$ as follows

$$
\dfrac{
\dfrac{\dfrac{\dfrac{0 \Vdash Tt \quad TA \Vdash TA}{0; T(t \to A) \Vdash TA}}{T(t \to A); 0 \Vdash TA}}{T(t \to A); Tt \Vdash TA}
\quad
\dfrac{\dfrac{\dfrac{0 \Vdash Tt \quad FA \Vdash FA}{0{:}\, FA \Vdash F(t \to A)}}{FA; 0 \Vdash F(t \to A)}}{FA; Tt \Vdash F(t \to A)}
}{Tt \Vdash T((t \to A) \to A)}
$$

The rule $0\mathbf{ij}$ is valid in **EW** because of the validity of $A \circ t \to t \oplus A$ (as $\vdash A \leftrightarrow A \circ t$ and $\vdash A \to {\sim}(t \to {\sim} A)$).

To model **C** and **CK**, we add the appropriate structural rules, and identify ':' and ';' because they collapse in the logic, by having exactly the same combinatorial properties. So, we have managed to sketch the proof of the following lemma.

LEMMA 7.13 *If* $X \Vdash A$ *in* **X**, *then* $X \Vdash A$ *in* **GX** + Cut.

Given this, we prove cut elimination just as before. Nothing much is made more difficult — it just becomes more tedious, because of the extra rules. We will give the changes in the definitions, and sketch the differences in the proofs, in order to prove cut elimination in the extended case.

7.5 Cut Elimination, Part 2

In the definitions we replace references to formulae to refer to signed formulae. So, we have

Definition 7.9 If X is a bunch, and $\mathcal{A}$ is a (possibly empty) set of occurrences of the signed formula SA in X, then $X(Y|\mathcal{A})$ is the bunch given by replacing those occurrences of SA by the bunch Y.

The mix rule is now:

$$\frac{X \Vdash S^1A \quad Y \Vdash S^2B}{Y(X|\mathcal{A}) \Vdash S^2B} \text{ Mix}$$

where $\mathcal{A}$ is a set of of occurrences of S^1A in Y. As before, the cut rule is given by the mix rule.

In order to show that **GX** + Mix gives us no more than **GX**, we need to show that proofs with mixes in them can be rewritten without them. Again, we will show that a proof with a mix in it can always have the mix rule either 'pushed back' to earlier in the proof, or done away with altogether. Recall Definition 7.2 concerning inferences and rules.

Definition 7.10

- Let Inf be an inference which is an instance of the rule Ru. A *conclusion parameter* is a signed formula occurrence in the conclusion of Inf which is not the newly introduced "principal signed formula" of one of the rules, nor the bunch Z that is introduced by eK or **K**. A *premise parameter* is a signed formula occurrence in a premise of Inf which "matches" a conclusion parameter in Inf.

Definition 7.11 Let Inf be an inference. For example,

$$\frac{X_1 \Vdash S^1A_1 \ \cdots \ X_n \Vdash S^nA_n}{X_{n+1} \Vdash S^{n+1}A_{n+1}}$$

and let $\mathcal{B}$ be a set of conclusion parameters, all of which occur in X_{n+1}. For each i, let $\mathcal{B}_i$ be the set of premise or conclusion parameters in X_i matching at least one of those in $\mathcal{B}$. For any bunch Y, define $Inf(Y|\mathcal{B})$ as

$$\frac{X_1(Y|\mathcal{B}_1) \Vdash S^1A_1 \ \cdots \ X_n(Y|\mathcal{B}_n) \Vdash S^nA_n}{X_{n+1}(Y|\mathcal{B}_{n+1}) \Vdash S^{n+1}A_{n+1}}$$

So, $Inf(Y|\mathcal{B})$ is given by replacing each of the parameters matching those in $\mathcal{B}$ by Y. The rule Ru is said to be *left regular* if for every inference Inf and every set $\mathcal{B}$ of conclusion parameters occurring in the antecedent of Inf, if Inf is an instance of Ru, so is $Inf(Y|\mathcal{B})$.

To define "right regularity," we need a little more subtlety. Let Inf be as given above. Let SC be a signed formula, and Y a bunch. Let $\mathcal{B}$ be the unit set of some signed formula occurrence in Y. Set S_i to be either $Y(X_i|\mathcal{B}) \Vdash SC$ or $X_i \Vdash S^iA_i$ according as S^iA_i does or does not match the conclusion parameter $S^{n+1}A_{n+1}$. Then $Inf(SC, Y, \mathcal{B})$ is defined as

$$\frac{S_1 \ \cdots \ S_n}{S_{n+1}}$$

So, $\mathrm{Inf}(SC, Y, \mathcal{B})$ is given by systematically replacing parameters matching those in the consequent of the conclusion by SC, and simultaneously embedding the antecedents of these sequents in a larger context, represented by the occurrence of a formula occurrence in Y.

A rule Ru is *right regular* if whenever it has Inf as an instance, it has $\mathrm{Inf}(C, Y, \mathcal{B})$ as an instance, for all C, Y and $\mathcal{B}$ a unit set of a signed formula occurrence in Y.

As before, we can check each instance.

LEMMA 7.14 *Every rule of* **GX** *is both left and right regular.*

Proof: By inspection. Each 0-rule (including $0\mathbf{ij}$) is left regular as we are not substituting for 0, which is a bunch, and not a formula. ◁

To prove that mix can be eliminated from our proofs, we need to define give a rigorous definition of what it is to 'push' a mix backward in a proof. To do this, we need some kind of handle on how far down a mix comes in a proof. This means defining the concept of rank for signed formulae.

Definition 7.12 Let Der be a derivation, with S as its final sequent. Unless S is an axiom, take Inf to be the inference of which S is the conclusion, and take $\mathcal{A}$ to be a set of signed formula occurrences in S. We define the *rank of* $\mathcal{A}$ *in* Der as follows:

- If $\mathcal{A}$ is empty, its rank is 0.
- If $\mathcal{A}$ is non-empty, and contains only formula occurrences that are introduced by e-weakening, $\mathbf{K}$, or a logical rule, or if S is an axiom, the rank of $\mathcal{A}$ is 1.
- Otherwise, let $S_1, \ldots, S_n$ be the premises of Inf, for each i let Der_i be the subproof terminating S_i, and let $\mathcal{A}_i$ be the (possibly empty) set of premise parameters in S_i that match at least one member of $\mathcal{A}$ (at least one $\mathcal{A}_i$ will be non-empty, granted that we are in this case). Let r_i be the rank of $\mathcal{A}_i$ in Der_i. Take the rank of $\mathcal{A}$ in Der to be $1 + \max_{1 \le i \le n} r_i$.

In words, the rank of a set of signed formula occurrences in the definition is the length of the longest path upward in the derivation of matching signed formulae in that set, plus one. (And it's zero if the set is empty).

We need just define the complexity of a signed formula, as we will be performing an induction on just this complexity. The new notion is parasitic on the old. The *complexity* of the signed formula SA is the complexity of the formula A. In other words, SA is more complex than $S'B$ just when A is more complex than B.

Now we can proceed to the mix elimination proof. The general procedure is the same as before, and some of the details are unchanged. In this proof we will see only the wrinkles introduced by negation.

THEOREM 7.15 *If* $X \Vdash SA$ *in* **GX** $+ \mathrm{Mix}$ *then* $X \Vdash SA$ *in* **GX**.

Proof: Take sequent that can be proved in $\mathbf{GX} + \text{Mix}$ with a proof ending in a mix. We show that this sequent can be proved without that mix. Take the final mix and label the premises and the conclusion as follows:

$$\frac{[L]\quad X \Vdash S^1A \qquad [R]\quad Y \Vdash S^2B}{[C]\quad Y(X|\mathcal{A}) \Vdash S^2B}\ \text{Mix}$$

Let k be the rank of the consequent occurrence of S^1A in the subproof ending in L, and let j be the rank of the occurrences $\mathcal{A}$ in the subproof ending with R. The rank of the mix is $k + j$.

As before, we use the double induction from before. The outer induction hypothesis is that all mixes of signed formulae less complex than SA are eliminable. The inner induction is that given SA, all mixes of rank less than the mix we are considering are eliminable.

Given the mix displayed above, five cases exhaust the options, as before.

Cases 1 to 4. Exactly as in the proof for decidability of the positive systems.

Case 5. $k = 1$, $j = 1$, R is not an axiom, and it is not given by eK. So, R is given by a logical rule, and $\mathcal{A}$ is the unit set containing the signed formula occurrence newly introduced into Y, and the displayed A in L is *also* given by a logical rule. These logical rules must introduce the same connective. We will rehearse each case, because they are different enough from the positive cases to merit attention.

5.1a $SA = T(A_1 \circ A_2)$ or $SA = T(A_1 \wedge A_2)$. The proof is as before, the rules are no different to those in the positive system.

5.1b $SA = F(A_1 \wedge A_2)$. Then L is either of the following

$$\frac{X \Vdash FA_1}{X \Vdash F(A_1 \wedge A_2)} \qquad \frac{X \Vdash FA_2}{X \Vdash F(A_1 \wedge A_2)}$$

Whichever it is, R must be

$$\frac{X(FA_1) \Vdash SB \quad X(FA_2) \Vdash SB}{X(F(A_1 \wedge A_2)) \Vdash SB}$$

Given this, we can push the mix back beyond the left inference, reducing the complexity of the mix signed formula.

5.1c $SA = F(A_1 \circ A_2)$. Then L and R must be

$$\frac{X{:}\,TA_1 \Vdash FA_2}{X \Vdash F(A_1 \circ A_2)} \qquad \frac{Y(FA_2) \Vdash SB \quad Z \Vdash TA_1}{Y(Z{:}\,F(A_1 \circ A_2)) \Vdash SB}$$

5.2a $SA = T(A_1 \vee A_2)$. Proof as in the positive systems.

5.2b $SA = F(A_1 \vee A_2)$. Dual to the case for $T(A_1 \vee A_2)$. We leave this case for the reader.

5.3a $SA = T(A_1 \to A_2)$. Then L is

$$\frac{TA_1; X \Vdash TA_2 \quad FA_2; X \Vdash FA_1}{X \Vdash T(A_1 \to A_2)}$$

and R is either one of the following two inferences.

$$\frac{Z \Vdash TA_1 \quad Y(TA_2) \Vdash SB}{Y(Z; T(A_1 \to A_2)) \Vdash SB} \qquad \frac{Z \Vdash FA_2 \quad Y(FA_1) \Vdash SB}{Y(Z; T(A_1 \to A_2)) \Vdash SB}$$

The case where R is the first of the two is just as before. If it is the second, then by hypothesis, the proof of $Z; X \Vdash FA_1$ (which is mixed from $FA_2; X \Vdash FA_1$ and $Z \Vdash FA_2$) is eliminable, and we can then eliminate the mix that uses this and $Y(FA_1) \Vdash SB$ to give $Y(Z; X) \Vdash SB$, as we desired.

5.3b $SA = F(A_1 \to A_2)$. Then L and R are

$$\frac{X_1 \Vdash TA_1 \quad X_2 \Vdash FA_2}{X_1 : X_2 \Vdash F(A_1 \to A_2)} \qquad \frac{Y(TA_1 : FA_2) \Vdash SB}{Y(F(A_1 \to A_2)) \Vdash SB}$$

Then as before, this mix is eliminable by replacing it with two. One mixing X_1 into TA_1, and another mixing X_2 into TA_2.

5.4 $SA = Tt$. Then the left sequent is $0 \Vdash Tt$, and the right is $Y(Tt) \Vdash SB$, which was deduced from $Y(0) \Vdash SB$. The conclusion of the mix is $Y(0) \Vdash SB$, which we have already shown. So, the mix is eliminated. (There is no introduction rule for Ft, so this case doesn't appear.)

5.5 The mix formula is $S{\sim}A$. Then L and R are

$$\frac{X \Vdash \overline{S}A}{X \Vdash S{\sim}A} \qquad \frac{Y(\overline{S}A) \Vdash SB}{Y(S{\sim}A) \Vdash SB}$$

where $\overline{S}$ is the opposite sign to S. Then clearly the mix can be pushed back to $\overline{S}A$, a formula of lower complexity.

As before, each possible mix can be eliminated. So, our theorem is proved. ◁

7.6 Decidability, Part 2

We can use this cut elimination result to prove the conservative extension results for the negation systems too. Recall Definition 7.5 — this still holds in the negation-added systems.

LEMMA 7.16 *All proofs in* **GX** *have the subformula property.*

Proof: By inspection, as before. ◁

LEMMA 7.17 *No proof of an i-only-sequent is insane.*

Proof: As before. ◁

LEMMA 7.18 *If an i-only-sequent has a confused proof, it also has a proof that is not confused.*

Proof: As before. ◁

So, as before, we have our conservative extension result.

COROLLARY *Each sequent in the intensional vocabulary is provable in a purely intensional vocabulary in* **GX**. *So,* **GX** *conservatively extends* $\mathbf{GX}_{\to\circ\leftarrow t}$

We also have the primeness result, by way of cut elimination.

THEOREM 7.19 $\vdash A \vee B$ *if and only if* $\vdash A$ *or* $\vdash B$.

Recall Definition 7.6, about reduced and semi-reduced bunches. We can use these definitions in our expanded context.

LEMMA 7.20 *Any provable reduced sequent has a semi-reduced proof.*

Proof: As before. ◁

The loose equivalents of a semi-reduced sequent are still those semi-reduced sequents from which the original sequent can be obtained by any of the structural rules available in the logic. We can the prove the following results, just as before.

LEMMA 7.21 *Every semi-reduced sequent has only a finite number of semi-reduced loose equivalents, and these can be effectively listed.*

LEMMA 7.22 *If* S *is of complexity* n *and* S' *is equivalent to* S, *then* S' *is of complexity at most* n.

Proof: The new structural rules keep complexity constant. They make no difference to the result. ◁

THEOREM 7.23 *Any of our logics* **X** *are decidable.*

Proof: The proof has exactly the same structure. By induction on the complexity of sequents. The base case is still valid. Working back in the proof, there are only a finite number of equivalents of any sequent we wish to prove. Once we've exhausted all of these, we only have less complex sequents to derive from, which are all decidable by hypothesis. ◁

We've shown that though adding negation certainly increases complexity (look at the size of the proofs!) it doesn't lead to undecidability. In that sense, the propositional systems are tractable.

The results work for **EW**, $\mathbf{L}^+$ and $\mathbf{LI}^+$ because of the way we have distinguished between 0 and t. On our account, 0 is a bunch, and so, substituting for 0 is not allowed for left and right regularity. So, we can add special properties of t to our logics by adding rules for 0 — like the 0-swap rules. These are not extra connective rules for t, but *structural* rules governing the behaviour of our structural 'connective' 0. In this way, the logics have identical rules for the connectives, and they only vary with respect to structural rules.

7.7 Validities

How do you model world validity in a Gentzen system? The method in not difficult — it is just like the case for natural deduction. We introduce a new bunch symbol W, which stands for an arbitrary world-like bunch. W will satisfy a number of structural rules, depending on what properties we like worlds to have. Then to show that $\Sigma \vdash_W A$, we look for a Gentzen proof of $W \Vdash A$, for which the axioms are among the identities, *and* the sequents $W \Vdash B$ for any $B \in \Sigma$. That is all. Does a Gentzenisation give us a handle on the decidability status of various kinds of world validity? Quite possibly. But this will have to be a matter for future work.

7.8 Note

[1] However, not all contraction-added systems that take $X; X$ to be different from X are undecidable. The distribution-rejecting system **LR** (given by taking away the distribution axiom from the Hilbert axiomatisation of **R**) is decidable [62, 160]. However, recent work by Urquhart shows that practically speaking, it is only barely decidable [164]. The decision procedure is unspeakably complex in general. For the conjunction, implication and fusion fragment of **LR** (which is the same as that fragment of **R**) the decision problem is *at least* exponential space hard with respect to log-lin reducibility. The only upper bound is given by Kripke's decision procedure, which is primitive recursive in the Ackermann function.

Chapter 8
Contraction-Free Applications

> ... the rest have worn me out
> with several applications ...
> — WILLIAM SHAKESPEARE *All's Well that Ends Well* [141]

8.1 Contraction and Indices

As we've seen, the ternary relational semantics for conditionals is formally powerful; it can be used to give a formal analysis of a wide range of formal systems. This is useful in and of itself. However, it does not amount to a *semantics* for our systems in a sense of assigning real meanings or interpretations to terms. For this we need to provide some kind of interpretation of the formal structures themselves. This is the task of the two chapters ahead of us. However, we must keep track of the direction of the work as a whole: we are *not* primarily interested in interpreting the simplified semantics as a whole. We are studying systems without contraction. Our aim is to consider interpretations of the ternary relational semantics which will allow contraction to fail.

EXCURSUS: For the current discussion, we don't care to pin down a *preferred* interpretation of the indices in the semantics. Of course this is an important task in some contexts. To establish how to best give a reductive account of indicative or subjunctive conditionals involves deciding whether this in best done terms of possible worlds, situations or something else. However, there is a wide range of conditional-like connectives modelled by the semantic structures of Chapters 5 and 6. Were these to turn out not to be useful in modelling natural language conditionals, they are certainly useful in modelling *something*. We shall see a number of applications of the semantic structures in this chapter and the next, where the indices are not worlds, but degrees of stretch, theories, actions, and situations. The things we learn in examining these structures will be useful in the study of natural language conditionals (which could do with a liberation from an approach that only considers 'possible worlds' as the indices of semantic evaluation) if only as an alternative against which to compare standard accounts, and an opportunity to see a wider range of semantic structures that can be used to model similar phenomena. This can only be helpful when confronted with a kind of conditional to model. The broader the range of tools available, the better. □

Consider how the absence of contraction shapes the semantics. Firstly, if **WI** is absent in our logics, we must have the possibility that $A \wedge (A \to B) \to B$ fail at the base index g. So, we have to allow the presence of an index x where $x \models A \wedge (A \to B)$ while $x \not\models B$. So, x must fail to be closed under *modus ponens*; x must fail to be *detached*.

For our structures this means that $Rxxx$ must fail. For were it to succeed, $x \models A$ and $x \models A \to B$ would deliver $x \models B$ by the evaluation clause for the conditional. So, the failure of **WI** brings with it the possibility of undetached indices, and the failure of $Rxxx$ (in general).

We have also seen that other postulates are contraction-related. In particular, **W**, **CSyll** and **S**. For the semantics, these postulates are delivered by the following rules:

W	$Rabc \Rightarrow R^2a(ab)c$
CSyll	$Rabc \Rightarrow R^2(ab)bc$
S	$R^2a(bc)d \Rightarrow R^3(ab)(ac)d$

W and **S** are properly stronger than **WI** in that adding them to a logic as strong as $\mathbf{DW}^+$ brings **WI** along with it. The case for **W** we have seen in Chapter 2, section 5. For **S** we note that

$$\big(A \wedge (A \to B) \to (A \to B)\big) \to \big((A \wedge (A \to B) \to A) \to (A \wedge (A \to B) \to B)\big)$$

is a substitution instance of the **S** axiom, and the theoremhood of $A \wedge (A \to B) \to (A \to B)$ and $A \wedge (A \to B) \to A$ gives the **WI** axiom as desired.

On the other hand, **CSyll** gives **WI** only in the presence of stronger axioms like $\mathbf{C}''$ and its dual. If $\vdash A \leftrightarrow (t \to A)$ we have

$$(t \to A) \wedge (A \to B) \to (t \to B)$$

as a substitution instance of **CSyll** so the biconditional gives us $A \wedge (A \to B) \to B$, the **WI** axiom.

Given this caveat, we have seen that **WI** is the lowest common denominator of contraction related moves. So for there to be a thoroughly contraction-free logic with a ternary relational semantics we must ask ourselves this question: why would there be undetached indices? How can we interpret the semantic structures in such a way as to make undetached indices feature? Once we answer this, we will have demonstrated a use for contraction-free logics, and specifically, for logics without contraction in *any* form (without **W**, **CSyll** or **S** as well as without **WI**). Thankfully for us there are a number of good answers to this question. We will consider four: three in this chapter, and the fourth in the next.

8.2 Vagueness[1]

Slaney has sketched out a case for the connection between contraction-free logics and a decent account of vagueness [144]. In this section we'll draw upon his work to motivate one interpretation of the semantics that gives rise to undetached indices.

However vagueness is to be dealt with, it is clear that the phenomenon gives rise to something like 'degrees of truth'. Almost as clearly, this idea has been abused badly, making many jump to the conclusion that truth comes linearly ordered in degrees between zero (the truly, precisely and determinately False) and one (the truly, precisely and determinately True), giving rise to something very much like Łukasiewicz's infinitely valued logic. This motivation seems to drive the fuzzy logic industry. Instead of jumping to this conclusion, we will examine the real cash value inherent in the idea of 'degrees of truth'.

What can we make of this idea? It seems to be motivated as follows: take a classic borderline case of a vague predicate. Standard examples are predicates like 'bald', 'old', 'heap', or even 'chicken',[2] but for the moment we will stick to an old favourite like 'red'. Take a borderline case of red, like a shirt I wear. The statement 'that shirt is red' may not be true, but it doesn't get things wrong in quite the same way (to the same degree) that the statement 'the surface of the moon is red' gets things wrong. You are not stretching the truth as far to count that shirt as red as you would be if you took the surface of the moon to be red.

Slaney himself gives a *prima facie* case for degrees of truth. It goes like this:

> 'Grass is green' is true if and only if grass is green. That is, the truth of 'Grass is green' marches together with the greenness of grass. But greenness is a matter of degree; so truth is a matter of degree. [145]

However, as Slaney notes, this argument needs a lot of fleshing out to deliver its validity. Richard Sylvan has noted that there are plenty of invalid arguments of the same form:

> The rich and the poor march together in this life ... The rich are well off, therefore so are the poor. Correlation is not in general a sufficient basis for property transfer. [157]

Instead of arguing further for the claim that there's something to the conception of 'stretching' the truth, we assume that *something* like that conception is right, and then see how far we can develop the idea, and what follows from it.

It is perhaps best to spend a little more time dispelling myths about a 'degrees of truth' approach to vagueness. The approach suffers from an exceedingly naïve treatment at the hands of almost all of its practitioners. Specifically, people often jump from the premise that truth has something to do with degrees to the conclusion that the degrees come neatly ordered in the interval $[0, 1]$. This is a large and unwarranted jump. Is it seriously contended that any statement is true to degree $0.9060939428\ldots$? Why this and not $0.9060939429\ldots$? Much more work would have to be done to conclude that every statement can be assigned a single determinate degree from the interval $[0, 1]$.

Determinate values are not the only problem for this approach. It is also a strong claim that these values can be *totally ordered*. This means that the degree to which "The grass on the outfield of the MCG is green" is true can be compared with the degree to which "Greg's Mazda 1300 is rather old" is true. Both seem *reasonably* true, but there is nothing to say that one is truer than the other. Yet this is what the standard brands of fuzzy logic dictate. Truth values are totally ordered, so the truth values of any two statements can be compared. This results in the rather odd looking

$$(A \to B) \lor (B \to A) \qquad \textbf{Ord}$$

ending up as a theorem. There is no rhyme nor reason for this. It is much more realistic to have a semantics in which truth is not only stretched, but in which the degrees of truth are only *partially* ordered. The degree to which you stretch the truth to take "The grass on the outfield of the MCG is green" to be true is more likely to be incomparable to the degree to which you stretch the truth to take "Greg's Mazda 1300 is rather old" to be true. Different statements can stretch the truth in different directions.

The 'degrees of truth' approach can be sanitised even further. To say that something is true to a certain degree does not have to be seen as saying that truth itself actually comes in degrees. Rather, it can be seen as endorsing the sensible view that vague predicates come with a certain degree of slack. For something to be a borderline case of red is only to say that you are not stretching the extension of red very far to include it. The degrees of truth can be seen as varying ways of stretching out the extensions of predicates. This preserves the point that truth can be stretched in different (orthogonal) ways. It also avoids the criticism that there has to be a precise, quantitative degree of truth for each statement. To see how this works, we must develop the account of degrees of stretch.

Consider a collection of degrees of stretch of the truth. For a degree x, and a claim A, $x \models A$ means that by stretching the truth (or extensions of predicates) out to x we make A true. It is important to understand what this symbolism means. The 'degree' x denotes a particular precise degree to which we might stretch the truth. A degree x will determine a precise extension for each predicate (and perhaps a precise anti-extension too, if we wish to model negation using the four-valued semantics). Given this view of degrees of stretch, the semantics of conjunction and disjunction are simple. It seems clear that $x \models A \wedge B$ if and only if $x \models A$ and $x \models B$. A degree of stretch will support a conjunction if and only if it supports both conjuncts. Dually, $x \models A \vee B$ if and only if $x \models A$ or $x \models B$. This is perhaps not so obvious, because you may wish to hold that a degree of stretch could support a disjunction without deciding *which* disjunct is true. For example, if degrees of stretch are analogous to epistemic states, then it is plausible to say that a disjunct 'takes you further' than its disjunction. However, in many of the traditional analyses of vagueness, such as the supervaluational account or even the epistemic account, degrees of stretch are truly prime. That is, they evaluate disjunctions in the usual manner. A degree in the supervaluational account is simply a classical evaluation — these have prime evaluations for disjunction. On the epistemic account, degrees are simply possible worlds which are epistemically accessible. These too treat disjunction in the standard fashion. So, it seems that we are not pushing things too far to take degrees of stretch to evaluate disjunction in the classical manner.

The heart of my proposal concerns the interaction of degrees of stretch with implication. For this, it is helpful to consider a typical case of the sorites paradox, with a strip of colour shading from canary yellow to scarlet red. Divide the strip into 10 000 pieces, so that each piece is indiscernible in colour from its immediate neighbours. Let Y_i be the claim "piece i appears yellow (under normal lighting conditions)." Then, Y_1 is true by the way things are set up, and each conditional $Y_i \rightarrow Y_{i+1}$, if not *true*, is certainly not stretching the truth very far at all. So far, there is nothing too odd about the proposal. The novelty comes with the next thought.

The thought is this: It is quite plausible to suppose that there is a degree of stretch x that satisfies $x \models Y_1$ and $x \models Y_i \rightarrow Y_{i+1}$ *for each* i. After all, we are inclined to believe such things. When presented with such a colour strip, I may say that if a colour patch appears yellow, then so does its immediate successor, given that I cannot discern any difference between them. However, it is stretching the truth a great deal further than this to suppose that $Y_{10\,000}$ is true, and so, we won't have $x \models Y_{10\,000}$.

What does this mean for this degree of stretch? An obvious implication is that the degree of stretch is undetached. For we have $x \models Y_1$, $x \models Y_1 \to Y_2$, $x \models Y_2 \to Y_3$, ... , $x \models Y_{9\,999} \to Y_{10\,000}$ while $x \not\models Y_{10\,000}$. The chain of inference by *modus ponens* must break down somewhere, so x is undetached. It is not closed under *modus ponens*.

This gives us a motivation for the failure of contraction in this context. In the presence of vagueness, the conclusion of some application of *modus ponens* may stretch the truth further than both of the premises do. If $x \models A \to B$ and $x \models A$, it may take more than x to stretch the truth out to get B. So, using conditionals at degrees of stretch may result in the truth being stretched. *Applying* the information at x to x itself results in another degree of stretch. Quite plausibly, one which stretches the truth more.

We can generalise this to arbitrary pairs of degrees, in order to get the semantics of conditionals in terms of degrees of stretch. Take $Rxyz$ to mean *applying* y *to* x *stretches the truth out no further than* z. From our example it follows that $Rxxx$ will sometimes fail.

Given such an account of the interactions of the degrees of stretch, how are conditionals modelled? If $y \models A \to B$, then if $x \models A$ and $Rxyz$ we have $z \models B$. Because the result of applying y to x (which must include B, at least) is contained in z. But conversely, if for each x where $x \models A$ and $Rxyz$ we have $z \models B$, then it's clear that y must support $A \to B$; because whatever x we choose, provided it supports A, the result of applying y to it supports B. The degree y is applied to A-supporting-indices results in B-supporting-indices. The degree y licences the transition from A to B, and so, it supports $A \to B$. So, we have the standard modelling clause for conditionals in the ternary-relational semantics, and it makes sense on our interpretation: $y \models A \to B$ iff $x \models A$ and $Rxyz$ then $z \models B$.

Given all of this, a semantic structure will be a field of degrees of stretches of the truth, together with an evaluation that tells us which predicates are true of which objects at which degrees of stretch and a ternary relation which tells us how the degrees of stretch interact.

Once we are faced with a relation like R in a semantics, we must ask ourselves what it means. A number of answers are possible. If we were more specific about the nature of the degrees of stretch, an interpretation of R may arise naturally from that conception. However, another way of interpreting R is possible. This interpretation takes our intuitive claim that R denotes 'application of conditional information' at face value. At base, $Rxyz$ just means 'for each A and B where $x \models A$ and $y \models A \to B$ then $z \models B$.' Fundamentally, R gets its properties from the truths of conditionals, *and not vice versa*. This means that the semantics of conditionals is not a reductive semantics, explaining the semantic content of sentences in terms of primitives that do not use concepts such as conditionality. An example from elsewhere in the literature will be helpful at this point.

Consider Lewis' semantics for counterfactual conditionals. This semantics uses a notion of 'closeness' of possible worlds. Clearly, Lewis must give some kind of account of what it means for possible worlds to be close or far apart. And this he does. After considering various puzzle cases, which show that various intuitive notions of similarity will not do, the fundamental answer is this: Similarity of possible worlds is fundamentally a matter of which counterfactuals are true. Does this mean that his semantics is useless? Of course not. It suggests a connection between the semantic evaluation of counterfactuals and a relation between possible worlds. It provides a regimentation of our intuitions about counterfactuals. Counterfactuals can now be treated uniformly by means of a nearness relation between possible worlds. This gives us some purchase on the valid arguments involving counterfactuals.

This semantics for conditionals in the presence of vagueness can be seen as providing a similar connection, this time between conditionals, and a relation between degrees of stretch of the truth. Now the relation is ternary, and the degrees of stretch are not all closed under *modus ponens*, for reasons we have seen. Our analysis codifies the truth of conditionals in terms of the relation of application between degrees of stretch of the truth. This notion, to be sure, is in turn to be cashed out in terms of conditionality, so the account does not give the semantics for conditionals in a purely conditional-free form. (Unless some reductive account of the application relation is later given.) But this is no surprise, and no fault of a semantic scheme. The clauses for conjunction and disjunction in our semantics (and in any other widely used semantics) are not problematic. But they are not reductive. They give an account of the behaviour of conjunction and disjunction in terms of conjunction and disjunction. Our formal semantics can be interpreted without giving a reductive account of the ternary relation R of application. Just as there is a vague intuitive idea of 'closeness' of possible worlds which gives the Lewis style semantics for counterfactuals its usefulness, I submit that there is an intuitive idea of 'application' between degrees of stretch, which does the same for this semantics. Perhaps there can be a reductive account of application, but the present point is that there need not be such an account for the semantics to be interpreted in the intended fashion.

To get the logic of conditionals in the presence of vagueness we must determine the properties of R. All we know for sure is that $\mathsf{R}xxx$ will fail on occasions. To determine the other properties of R there seems to be no other route than considering the connections between vagueness and conditionals. However we decide to make these choices (and there is no reason to say that there is a unique choice to be made, given the variety of ways conditionals are used) we have a sensible account of conditionality and the degrees to which the truth can be stretched, without falling into the problems of the naïve approach to degrees of truth. If we opt with Slaney to make R commutative, associative and hereditary (that, satisfying **B**, **C** and **K**) we end up with the logic **CK** (in at least the positive part). This is appealing for a number of reasons. Firstly, we have kept the logic as strong as possible while still retaining our intuitions about some indices being undetached. Secondly, it as it is very close to $\mathbf{Ł}_\infty$, which is favoured by the fuzzy logic community. **CK** has all of the advantages of lacking the unmotivated 'theorems'

$$(A \to B) \vee (B \to A)$$
$$\big((A \to B) \to B\big) \to \big((B \to A) \to A\big)$$

Thus we get quite near the work of the fuzzy logic industry, without having to make any strange commitments concerning the precise degree of truth of each statement.[3]

For a more complete semantics we also need to consider the behaviour of conjunction, disjunction and negation. The first two seem unproblematic. The standard clauses from Chapter 5 make sense under this interpretation. Negation is more difficult. Unless a meaningful interpretation can be made of the Routley-Meyer $*$ in this context, it appears that some kind of three-valued or four-valued account of negation is more plausible. Clearly we do not want to give negation a Boolean modelling (demanding that $x \models -A$ if and only if $x \not\models A$ for each degree x) because this would collapse any notion of indices being stretched and supporting more as the truth is stretched more. There would be no 'gaps' to be filled as the 'truth' gets expanded. However, as negation is not the central topic of this chapter, let alone this thesis, we must leave this discussion for another time.

So, we have an intuitively pleasing picture motivating contraction-freedom and undetached indices. Before we leave the matter, we had better work through some of the implications of such a semantic account.

In our semantic story, strong validity amounts to truth preservation at all indices. As indices in our structure are precise objects (whether or not $x \models A$ has a precise answer for any fixed x and A) strong validity is a precise notion. This is good, because it seems that logical validity is not a vague notion. Similarly the notions of weak validity $\vdash$ and worldly validity $\vdash_W$ are also precise (at least, given that the set W of worlds is a precise set). So, we are not faced with problems in interpreting validity.

However, despite the precise nature of the model structures, we have not excised vagueness from the realm of truth. This is also a good thing, because to do so would be to get rid of the very thing we were trying to account for. Rather, vagueness is preserved in that the matter of *which* degree of stretch is the actual one is vague.

To elaborate, we have a model in which the degrees are all the possible degrees of stretch. These are all precise objects. Each degree determines the extensions (and perhaps the anti-extensions, depending on your account of negation) of predicates exactly. In other words, the model structure is determinate. But this does not mean that truth is not vague. Instead, the question of which degree of stretch reflects the truth is vague. It is not determined by the model structure itself. A degree x is the actual degree of stretch just in case

$$\text{For each } A,\ x \models A \text{ if and only if } A \text{ is true}$$

If there is vagueness with respect to the truth of A, for some range of xs, whether or not it is the actual degree will be vague too. Of course, there are other restrictions on the actual degree of stretch. We know that any degree of stretch which has a hope of being the actual degree will be worldlike: It must be closed under *modus ponens*, because *modus ponens* is valid (in the sense of being truth preserving in the actual world.[4])

This a pleasing phenomenon, because it reflects a respected semantic account of vagueness, using *supervaluations*. Our degrees of stretch are similar to supervaluations, in that the (vague) truth is modelled as being indeterminate among a set of (precise) valuations. However, our account differs from the standard supervaluational account by having a different way of evaluating the conditional. Here we take the conditional to tell us something of relationships between degrees of stretch. Something that is impossible on the standard supervaluational approach. This is a helpful distinction, because it can pay its way in explanation. Consider the example from before. The conditional

$$Y_1 \wedge (Y_1 \to Y_2) \wedge (Y_2 \to Y_3) \wedge \cdots \wedge (Y_{9999} \to Y_{10\,000}) \to Y_{10\,000}$$

fails to be logically true on our account, as not every degree of stretch that satisfies the antecedent also satisfies the consequent. This makes sense, as it doesn't stretch the truth very far at all to admit the antecedent, but it does to admit the consequent. However, the related argument from Y_1, and each $Y_i \to Y_{i+1}$ is truth preserving at worlds, because if Y_1 and each $Y_i \to Y_{i+1}$ is true at the actual degree of stretch (whatever that might be) then so is $Y_{10\,000}$. Thus, we must say along with other accounts of the sorites paradox that not all of the conditionals $Y_i \to Y_{i+1}$ are true. However, we can retain the intuition that they are very close to being true, and that $Y_{10\,000}$ is very far from being true, by means of degrees of stretch and their interaction with conditionals. So, vagueness motivates the rejection of contraction on one interpretation of the semantics.

The next two applications for undetached indices are more sketchy, and they do not use the simplified semantics of this thesis, but rather, the operational semantics for relevant logics due to Fine [42]. For this we need to take a little detour to introduce Fine's work.

8.3 Fine Semantics

We need not go into this kind of semantic structure in any depth. For that, the reader is referred to Fine's paper [42]. For us, all we really need know is that it is a close cousin of the ternary relational semantics. Instead of a ternary relation, however, Fine uses a binary operation (which we write as concatenation) and the inclusion relation $\sqsubseteq$. Where we had $Rxyz$, Fine has $xy \sqsubseteq z$. The index xy can be thought of as y applied to x. (Though beware! For Fine, xy is x applied to y. Recall that we are reversing the order of R and fusion (and Fine's application operation). Keep this in mind when reading Fine's work.) So, we have $y \models A \to B$ if and only if for each x where $x \models A$, $xy \models B$. For stronger logics, you add conditions to the application operation, just as you add conditions to R in the ternary relational semantics.

This is a smooth presentation of the semantics in many respects. Generally, a binary operation is nicer than a ternary relation. The difficulty comes in the interpretation of disjunction. To see the problem, consider indices x and y where $x \models A$, and $y \models A \to B \vee C$. In many of our logics, we may have $y \models A \to B \vee C$ without $y \models A \to B$ or

$y \models A \to C$. Let this be one of those occasions. Now $xy \models B \vee C$ by the interaction of application and the conditional. Now, does $xy \models B$ or does $xy \models C$? There seems to be no way to decide. In fact, there is no reason that either must hold. If xy is in some sense a *minimal* index, containing just what you get from x by applying to it the conditionals given by y, then there is no reason that either B hold at x or that C hold at x. For this reason, disjunction cannot be evaluated classically in the Fine semantics. We must be more subtle. It turns out to be fairly simple in practice. Some indices are taken to be *prime*. These are the indices at which disjunction works classically. Then a disjunction is true at an index just when at each prime index that contains the original index, one of the disjuncts is true. In other words, a disjunction is true at an index if for any way of filling the index out to a prime index, one of the disjuncts must be made true.

Other than this modification, the Fine semantics is just like the ternary relational semantics. There is a logic index g such that $xg = x$ for each x, and strong validity is preservation at g. Contraction fails if there is an index x where $xx \not\sqsubseteq x$. This is enough of the Fine semantics for us to consider two possible applications.

8.4 Theory Commitment

Contraction-free logics also arise in the context of Lance and Kremer's work on theory commitment. We use the Fine operational semantics to model theories and commitment. We read $x \models A$ as 'x is committed to A,' where x is a theory. Then it is plausible to have

$$x \models A \wedge B \text{ iff } x \models A \text{ and } x \models B$$

To interpret the conditional Kremer and Lance urge us to, read $A \to B$ as

> a theory's commitment to A is [amongst other things] commitment to B [64]

Then it is most plausible to have $x \models A \to B$ iff for each y, if $y \models A$ then $x \models (y \models B)$ (thinking of $\models$ as a connective for a moment). In other words, if some theory is committed to A, then x is committed to that theory being committed to B, because x holds that commitment to A is in part, commitment to B. To make the converse plausible, we need that class of theories be large enough — that if x isn't committed to $A \to B$ then there must be a theory y that is committed to A but x is not committed to y being committed to B. It seems that if we consider the class of *all* theories (over a limited domain, perhaps) then this will be satisfied.

Now, the crucial part of this analysis of commitment and the way they are combined by the conditional $\to$ is the following jump: the collection of facts that x is committed to y being committed to is itself a theory. We'll call this theory yx. Then the evaluation clause for conditionals is simply the clause from Fine's operational semantics.

This simplification of the analysis makes a great deal of sense. One theory's commitments about another theory's commitments is bound to have the right kind of coherence to be a theory itself. It may not be a particularly rich theory, but it will be a theory nonetheless. Once this is admitted, and if we admit a special theory 1 — the *true* theory — and we add a relation $\sqsubseteq$ of theory containment, we can quickly establish a number of plausible conditions on how theories relate to each other:

$$x(yz) = (xy)z \quad 1x = x = x1 \quad \text{if } x_1 \sqsubseteq x_2 \text{ and } y_1 \sqsubseteq y_2 \text{ then } x_1y_1 \sqsubseteq x_2y_2$$

For the first, $x(yz) \models A$ iff $(yz) \models (x \models A)$ iff $z \models (y \models (x \models A))$ iff $z \models (xy \models A)$ iff $(xy)z \models A$. So, $x(yz)$ and $(xy)z$ are indiscernible with respect to their commitments, and so, are identical for the purposes of our semantics. For the second, if x is committed to the truth being committed to A, then x is committed to A, and so, the truth is committed to x being committed to A; and *vice versa*. The final condition is a simple fact about monotonicity.

Now, on this interpretation, **WI** amounts to the condition that $xx \sqsubseteq x$. That is, if x is committed to x being committed to A, then x is committed to A. Clearly, this is a desirable situation. A theory ought to be right about its own commitments. However, theories being the fallible things that they are, we have no assurance that this is the case. A theory may be committed to it being committed to A, without being in fact committed to A. It could be wrong about its own commitments — because of a faulty account of what commitment amounts to, or for some other reason. Whatever it is, if our account of theories is to be general, it is plausible to reject the postulate that $xx \sqsubseteq x$.

This motivates the rejection of $A \wedge (A \to B) \to B$ too. For $1 \not\models A \wedge (A \to B) \to B$ we must have some x such that $x \models A$, $x \models A \to B$ and $x \not\models B$. This is quite possible, for x may be committed to $A \to B$, and committed to A without *thereby* acquiring commitment to B. The most that x accrues is commitment to commitment to B, as it is committed to 'commitment to A is in part, commitment to B' and it is committed to A. It may in fact be false that commitment to A is in part, commitment to B, so x's commitment to A may not bring with it commitment to B. Here we see another place in which undetached indices have a place in a semantic structure. Possibly.

There is a deep objection to this claim that $xx \sqsubseteq x$ fails. Consider the case in which you are arguing with a friend. She shows you that you are committed to A, and that you are committed to $A \to B$. It is very hard to resist the deduction to the conclusion that you are *thereby* committed to B. She has shown how your commitments lead to B, and it does seem that you are thereby committed to B, at least until you reject one of your commitments to $A \to B$ and A. In contexts of reasoning about theories, we seem to use *modus ponens* with equanimity.

So, we leave our appeal to logics of commitment here. Lance leans towards systems without contraction[5], but as we have seen, there are some objections to their use in this context. It is up to the reader to see whether there is a use for undetached theories, and so, whether this account of the conditional has a use for contraction-free logics.

EXCURSUS: Although Lance and Kremer do not do this, it is possible to use the Fine interpretation of disjunction to model disjunction in the case of theory commitment. Take a limited class of theories to be the special *prime* theories. For theories x in this class, $x \models A \vee B$ iff $x \models A$ or $x \models B$. These are particularly rich theories which are able to ground each disjunction in a disjunct. The truth is one such theory; any other possible world provides a prime theory, and the empty theory is also (trivially) prime. But not all theories are so blessed to this extent. The theorems of classical logic form a theory, and this class certainly isn't prime. For non-prime theories, we can evaluate disjunction as Fine does. For such a theory x, it is committed to $A \vee B$ just when each prime theory containing x is either committed to A or committed to B. That is, there is no way of filling x out to a prime theory avoiding both A and B. While this is not particularly enlightening about the nature of disjunction, it seems to be an adequate analysis. (But then, this is about as much as one would hope for. Is the clause for conjunction any more enlightening? Extensional conjunction and disjunction are so basic that we can't expect much enlightenment from their respective clauses in a formal logic.) □

8.5 Actions

For a brief sketch of another application of the semantic structures we've seen, consider actions. As I perform actions of different types, I change the world around me. Consider a field of possible action types, and a binary operation of composition on action types. If x and y are actions types, then xy is the action type: x and then y. Composition is associative, and has the empty action as the left and right identity. It is not idempotent. Giving away \$50 and then giving away \$50 is not the same action as giving away \$50.

We can interpret the ternary relational semantics appropriately. The clause $x \models A$ becomes: as a result of x, A. Then the conditional clause:

$$x \models A \to B \text{ iff for each } y \text{ where } y \models A,\ yx \models B$$

has an interesting interpretation. It indicates a kind of transition from one state to another, by means of an action. For example, 'if I have \$50, I'll give away half of it'[6] is supported by an action if for each action that results in me having \$50, performing both of these actions will result in my giving away half of that \$50.

The two conjunctions have sensible interpretations in this context. Extensional conjunction is standard. An action x supports $A \wedge B$ if and only if it supports both A and B. Intensional conjunction is different. '$A \circ B$' is 'A and then B.' This follows from the modelling condition for fusion.

$$x \models A \circ B \text{ iff there are } y, z \text{ where } yz \sqsubseteq x,\ y \models A \text{ and } z \models B$$

Then we can expect *modus ponens* to come out in the form

$$A \circ (A \to B) \to B$$

since if $x \models A \circ (A \to B)$ then for some y and z, $y \models A$, $z \models A \to B$ and $yz \sqsubseteq x$. This means that $yz \models B$, and as $yz \sqsubseteq x$, $x \models B$ too, as desired. (This raises an interesting question: what does something like '$w \sqsubseteq v$' mean? There is a simple answer: $w \sqsubseteq v$ if and only if every action of type v is also an action of type w. Then if $w \models A$ we *must* have $v \models A$ too.)

However *pseudo modus ponens* fails. That is, we can have actions x where $x \models A \wedge (A \to B)$ while $x \not\models B$. The reason for this is that to use the conditional $A \to B$ supported by x, we need *another* action that supports A and then their composition will support B. This is perhaps hard to imagine, but there are concrete examples. Firstly, let's see what the semantics dictates. If $x \models A \wedge (A \to B)$, then by construction, $xx \models B$. So, this is one way to get the conclusion B from the action x. Just repeat the action: once to get the antecedent, and the next to get the conclusion. How does this work in practice? Consider my example conditional: "If I have \$50, I give away half of it." Perhaps an action of mine involves the giving away of \$25 of each \$50 I have. And also, *as a result of this* someone else gives me \$50. My action is still one which supports the conditional "If I have \$50, I give away half of it," because conjoined with the prior actions which resulted in my having \$50, I gave away the required amounts. (And by hypothesis, it would have resulted in me giving away \$25 of each other \$50 I would have had, in other circumstances, had other antecedent actions occurred.) However, another consequence of my action was my getting another \$50. It would be *another* action to give away half of this \$50. Clearly, if I were to perform an action of that type again, I would give away half of this \$50. However, I need not do this for my original action to be one which supports "If I have \$50, I give away half of it." Actions too may fail to be closed under *modus ponens*.

The conditional indicates a state-transition; it models the result of action composition, not conditionality within a single action. (This could be modelled by an intuitionistic clause: $x \models A \supset B$ iff for each $y \sqsupseteq x$, if $y \models A$ then $y \models B$, or some such thing.)

We will leave the task of fleshing out this interpretation, and fitting it into the rest of the literature on the topic of actions and state transition, for another occasion. We'll end this section by pointing out the similarity between this and some programmatic remarks by Girard on interpretations of linear logic.

> Classical and intuitionistic logics deal with *stable* truths:
>
> If A and $A \Rightarrow B$, then B, *but* A *still holds.*
>
> This is perfect in mathematics, but wrong in real life, since real implication is *causal*. A causal implication cannot be iterated since the conditions are modified after its use . . . For instance, if A is to spend \$1 on a pack of cigarettes and B is to get them, you lose \$1 in the process, and you cannot do it a second time. [52]

The semantic structures we have seen show us a way of modelling the kind of phenomena that Girard is concerned with in the natural setting of our semantic structures.[7]

8.6 Coda

From all of this we have three interpretations which give rise to undetached indices. The contraction-free perspective is fruitful, and can shed light on a wide range of problems, over and above the paradoxes of self reference (which were, after all, the initial motivations for this work). These applications do not explicitly lend credence to the project of giving an account of the paradoxes with contraction-free logics, unless vagueness, theory commitment or action are somehow implicated in the paradoxes. As far as I can see, they are not. These considerations do not give direct backing to the contraction-free account of the paradoxes. Instead, their backing is more subtle. The examples in this chapter have shown us that contraction-freedom naturally arises in a number of places. Once we open our eyes to the possibility that decent conditional operators might reject *pseudo modus ponens*, we see that there are many possibilities. There is room for much more study of these interpretations and the logics to which they give rise. We'll go on to examine the connections between situation semantics and contraction-free logics.

8.7 Notes

[1]This section owes much to Slaney's paper on the subject [144], which alerted me to the connection between vagueness and contraction-free logics. This section is attempt to flesh out some of the insights in that paper, and to see how far they can go.

[2]Consider the history of chickens. Given current evolutionary biology, it seems that their development was reasonably gradual. Given a particular chicken, if you trace back its ancestry, you will not find a sharp borderline between chicken and non-chicken. So, unless chickens were spontaneously generated, or there have *always* been chickens, there must have been borderline cases of chickens somewhere in history.

[3]It has been reported to the author that in Japan, vacuum cleaners are now advertised as coming "with fuzzy logic." It can only be hoped that soon there will be a range of vacuum cleaners released that "won't do contraction," or "will deduce relevantly."

[4]Graham Priest and I have discussed the consequences of denying the validity of *modus ponens* and I must admit I find it *very* hard to comprehend what it means to abandon *modus ponens* in this sense. Of course, it is quite easy to model it in the semantics. We just abandon the requirement that $Rwww$ for all world-like indices w. The trouble comes from the motivation to abandon *modus ponens*. The only reason for this seems to be to escape the conclusion of the sorites paradox. However, you must be careful if you go down this route. For you do not just abandon *modus ponens* for a particular conditional. You must abandon *modus ponens* for every conditional for which the premises of the sorites argument as true. Take $\Rightarrow$ to be an entailment connective. Let $A \Rightarrow B$ mean 'the argument from A to B is valid'. In the sorites case, get enough true antecedent conditions A to make $Y_i \wedge A \Rightarrow Y_{i+1}$ true (or perhaps $A \Rightarrow (Y_i \Rightarrow Y_{i+1})$, if you're concerned about the difference). A will contain things like truths about the meanings of the terms, the background lighting conditions, and the observer. You can pack enough in to make the conditional true. Then *modus ponens* for $\Rightarrow$ (and the truth of A) is enough to get the sorites argument up and running again. It is possible to dispute *modus ponens* for $\Rightarrow$, but then it is not clear what is meant by validity if it is not truth preserving.

[5]Mentioned in conversation in July 1992.

[6]Using the quite natural 'if ... then ...' to stand for the transition conditional.

[7]And in fact, it's better than a naïve application of linear logic, because in our semantic structures we won't have $xy = yx$ for all actions x and y. This means that we don't have $A \circ B \rightarrow B \circ A$, which is exactly how we want it, under this interpretation. Commutative linear logic is inappropriate here.

Chapter 9
Information Flow

> We may say that all 'information'
> is conveyed in the nervous system
> in the form of coded arrangements of nerve impulses.
> — JOHN CAREW ECCLES *The Neurophysiological Basis of Mind* [38]

9.1 Introduction

John Perry, one of the two founders of the field of situation semantics, indicated in an interview in 1986 that there is some kind of connection between relevant logic and situation semantics.

> I do know that a lot of ideas that seemed off the wall when I first encountered them years ago now seem pretty sensible. One example that our commentators don't mention is relevance logic; there are a lot of themes in that literature that bear on the themes we mention. [13]

In 1992, in *Entailment* volume 2, Nuel Belnap and J. Michael Dunn hinted at similar ideas. Referring to situation semantics, they wrote

> ... we do not mean to claim too much here. The Barwise-Perry semantics is clearly independent and its application to natural-language constructions is rich and novel. But we like to think that at least first degree (relevant) entailments have a home there. [6]

In this chapter I show that these hints and gestures are true. And perhaps truer than those who made them thought at the time. I will elaborate some of Jon Barwise's recent work [9] in which he sketches a new account of conditionals and the flow of information, based on situation semantics. Then, I argue that doing this naturally motivates the ternary relational semantics of one of our favourite logics.

This is a worthwhile task, for many have found the semantics for relevant logics unclear and ill motivated. (Merely a "formal semantics" as opposed to a "real semantics," as some have said.) By showing that this semantics has a natural home within an independently motivated account of information flow, we blunt this objection. On the other hand, the arguments here can be seen as motivating distinctions Barwise wants to draw, such as serial and parallel composition of channels (getting ahead of ourselves a little), because these have notions been at home in the work on relevant logics for the last thirty years or so, as intensional and extensional conjunction.

At the very least, this work gives an account of the harmonies between two very different research programs. A stronger thesis (to which I'm sympathetic) is that when situation semanticists give an account of conditionals they ought to be doing contraction-free relevant logic (of a particular kind), and that relevant logicians can escape the "not a *real* semantics" criticism (sustained by B. J. Copeland [23] among others) by appealing to situation semantics.

9.2 The Phenomenon to Model

Before going further, it is important to pause and get a 'feel' for the parts of situation semantics relevant to us. It is grounded in the Austinian account of truth [8]. According to (the situation semantics interpretation of) this account, a felicitous declarative utterance of a sentence S makes a claim about a *situation*, s. That is, about a particular piece of the world. It claims that the situation s is of a particular kind, or *type*, say φ. The situation that the utterance is about is called the *demonstrative content* of the utterance. It is typically determined by particular conventions of the language in use, and other relevant facts. What these conventions and other facts are is an interesting issue, but one that won't detain us. The type that the utterance describes the situation to be (correctly or incorrectly) is the *descriptive content* of the utterance. This is determined by other conventions of the language in use, and other relevant facts. Again, what these are will not be dealt with here.

Barwise intends to give an account of how information can flow from one situation to another. How can the fact that s_1 is of type φ tell us that s_2 is of type ψ? How is it that information about one situation can give us information about another? (Or even different information about the same situation?) This, in a nutshell, is the phenomenon Barwise seeks to model.

In setting out his story of how this happens Barwise has a number of plausible facts about information flow in mind. He intends to show that his account explains how these facts come to be true. To state these facts we need a little notation and a little more on types. We write '$s : \varphi$' to denote the *fact* that the situation s is of type φ. So, '$s : \varphi$' is a noun phrase. If you like, it may be helpful to treat $s : \varphi$ as a piece of information, provided that you keep in mind that information may well be *mis*information. This contrasts with '$s \models \varphi$' which is the *statement* that s is of type φ. So, '$s \models \varphi$' is a sentence, and it occurs in sentence contexts. Secondly we help ourselves to some ways for getting types out of other types. Type conjunction can be defined in the obvious way. If $s \models \varphi$ and $s \models \psi$, then $s \models \varphi \wedge \psi$ and *vice versa*. Similarly, $s \models \varphi \vee \psi$ if and only if $s \models \varphi$ or $s \models \psi$.

EXCURSUS: This is not *quite* so clear as the case for conjunction. Perhaps there is a situation in which either $\varphi \vee \psi$ obtains, but in which neither φ or ψ obtains. Of course, if this situation is *actual*, then either φ or ψ will obtain in a larger situation that includes our original one; disjunction isn't truly odd. For example, take the situation (or course of events) of the first half of a chess game. It may be that in this situation, either black will win or she will lose, or she will draw, or the game will be finished without result. This may immediately follow from the rules of chess, and the other elements of the situation at hand. It would be odd to say that the first half of the chess game is truly classified by any of the disjuncts. (Even if determinism is true.) At this point in time, one disjunct will be true (we suppose), but the situation that supports it will be a larger one than just the first half of the game. Despite this, at this point we may remain undecided as to whether situations are prime, as nothing much hangs on the issue (yet). □

Now we can consider some principles of information flow. Barwise takes these to be the facts any theory of the flow of information must deliver.

1. Xerox Principle If $s_1 : \varphi$ carries the information of $s_2 : \psi$ and $s_2 : \psi$ carries the information of $s_3 : \theta$, then $s_1 : \varphi$ carries the information of $s_3 : \theta$.

This is so called by Dretske [26]. This principle seems sound, and we wish our formal systems of information flow to give some kind of account as to why this principle works.

2. Logic as Information Flow If the type φ entails ψ (in some sense) then $s : \varphi$ carries the information of $s : \psi$.

This principle states that logical entailment is a variety of information flow. It seems worthwhile, and ought to drop out of a sensible account.

The next two principles make use of conjunction and disjunction.

3. Addition of Information If $s_1 : \varphi$ carries the information of $s_2 : \psi$ and $s_1 : \varphi'$ carries the information of $s_2 : \psi'$, then $s_1 : (\varphi \wedge \varphi')$ carries the information of $s_2 : (\psi \wedge \psi')$.

4. Exhaustive Cases If $s_1 : \varphi$ carries the information of $s_2 : (\psi \vee \psi')$, $s_2 : \psi$ carries the information of $s_3 : \theta$ and $s_2 : \psi'$ carries the information of $s_3 : \theta$, then $s_1 : \varphi$ carries the information of $s_3 : \theta$.

Barwise takes another principle to be axiomatic, involving a notion of negation on types.

5. Contraposition If $s_1 : \varphi$ carries the information of $s_2 : \psi$, then $s_2 : \sim\psi$ carries the information of $s_1 : \sim\varphi$.

This condition is plausible only under the assumption that $s \models \sim\varphi$ if and only if $s \not\models \varphi$. Then we can reason as follows: $s_2 : \sim\psi$ tells us that it's not the case that s_2 is ψ and so, it's not the case that s_1 is φ (by the information link) and so, s_1 is $\sim\varphi$. This reasoning is clear. However, the intuitions that ground this reasoning are unsound. For the kinds of negation we wish to model, situations are incomplete. For example, the situation of the Battle of Hastings ought neither be classified as supporting Queensland's losing the 1992 Sheffield Shield campaign, or as supporting Queensland's not losing that campaign. This situation tells us *nothing* about the Sheffield Shield.

Furthermore, in *Situations and Attitudes* [12] situations are allowed to be inconsistent (though Barwise and Perry take it that no *actual* situation is inconsistent). So, in any sense of 'not' that matters to us,[1] we can't perform the reasoning above.

For most of this chapter we'll consider the negation-free part of the formalism proposed by Barwise, and only near the end will we consider a more plausible way to deal with negation.

These are some largely pretheoretic facts about information flow and situations. We will now sketch Barwise's formalism, which is intended to give us an account of these facts, and which will model the behaviour of information flow in general.

9.3 Channels and Constraints

Clearly conditionals have something to do with the flow of information from one situation to another. Many conditionals indicate some kind of 'information link' between situations. So, giving an account of conditionals in terms of situations is a plausible way of giving an account of information flow. The first thing to do to incorporate conditionals in this framework is to find the descriptive and declarative contents of a typical utterance of a sentence *if* S_1 *then* S_2, where S_1 and S_2 are sentence fragments. Barwise's account is that the descriptive content is some kind of *constraint* of the kind $\varphi \to \psi$, where φ is the descriptive content of the statement S_1 and ψ is the descriptive content of the statement S_2. This constraint relates situations classified by φ to situations classified by ψ. Note that these need not be the same situation. In the presence of a chess game, I may utter

If white exchanges knights on d5 she will lose a pawn

This has the constraint $\varphi \to \psi$ as its descriptive content, where φ is the situation-type in which white exchanges knights on d5, and ψ is the type of situation-type in which she loses a pawn. These may be different situations, because the first may only be the next few moves, and the loss of a pawn may be further along in the game.

Note that this conditional is plausibly true even if we don't have for *every* situation of kind φ a corresponding situation of kind ψ. The game *could have* been very different (it could end after white's exchange because of some unforeseen circumstance) but this possibility does not defeat the conditional in itself. It is merely one of the background conditions of the conditional that this possibility does not obtain. This will become important soon.

The rest of the answer of the question of interpreting the conditional is less easy. What is the demonstrative content of an utterance *if* S_1 *then* S_2? What bit of the world is the speaker talking about when making this claim? What does the speaker classify as being of type $\varphi \to \psi$? One answer to this is immediate. Whatever it is, it must be a situation, if utterances of conditionals are to be modelled in the same way as other declarative utterances. In our case, the situation is the chess game (or a part of it). We are talking about the player white, and what will happen if she exchanges knights on d5. So much is common sense. The demonstrative content of the utterance is a situation.

However, Barwise does not explicitly follow this approach. Barwise takes the line that the demonstrative content is an *information channel*, grounding a relationship between situations. These channels might be due to physical law, convention, or logical necessity. However they come about, they are classified by constraints. So, for every channel c there is a binary relation $\overset{c}{\mapsto}$ between situations, so that $s_1 \overset{c}{\mapsto} s_2$ means that c relates s_1 to s_2 as antecedent situation to consequent situation. In our example, if c is the channel classified by the utterance of the conditional about the chess game, then the antecedent situations will be next move situations, and the for each antecedent situation the related consequent situations will be the situations of further progress in the game after that move chosen by white. Then a channel c is classified by the constraint $\varphi \to \psi$ just when for each s_1 and s_2 where $s_1 \overset{c}{\mapsto} s_2$ and s_1 is classified by φ then s_2 is classified by ψ. (There are no prizes for guessing where you've seen *that* kind of relationship before.) Note that a channel may not relate *every* situation to another situation, as in our example of the chess game. Our channel classified may relate only situations that are a part of a normal continuing chess game, without giving us any information about situations that are not a part of abortive games. This will be important in what follows.

Barwise uses this notion of an information channel as the ground of all information flow. All information flow is mediated by (or relative to) an information channel, in the same way that for Barwise, all truth is supported by (or relative to) some situation.

For now we have enough to begin defining structures of situations and channels. We will formalise the notion of classification, and the information structures we will study in what follows.

Definition 9.1 An *information structure* $\langle S, T, C, \models, \mapsto \rangle$ is a structure made up of a set S of *situations*, T of *types*, C of *channels*, a binary relation $\models$ which relates both pairs of situations and types and pairs of channels and constraints and a ternary relation $\mapsto$ relating channels to pairs of situations. This structure must satisfy a number of further conditions:

- Types are closed under binary operations $\wedge$ and $\vee$. Furthermore, for each $s \in S$ and each $\varphi, \psi \in T$, $s \models \varphi \wedge \psi$ if and only if $s \models \varphi$ and $s \models \psi$, and $s \models \varphi \vee \psi$ if and only if $s \models \varphi$ or $s \models \psi$.
- For every $\varphi, \psi \in T$, the object $\varphi \to \psi$ is a *constraint*. The relation $\models$ is extended to channels and constraints in the way indicated: $c \models \varphi \to \psi$ if and only if for each $s_1, s_2 \in S$ where $s_1 \overset{c}{\mapsto} s_2$, if $s_1 \models \varphi$ then $s_2 \models \psi$.

In the rest of this section we will extend this definition in order to model the principles of information flow. So, we will have a complete picture of an information structure as Barwise describes it only at the end of this section, where we will see the complete definition in all its glory.

Excursus: The condition dictating which channels support which constraints is fundamental to the theory of constraints and channels. The left-to-right part of the biconditional in the condition is clear. If a channel relates s_1 to s_2, (for example, white's move in this chess game, related to the next few moves) then the fact that the channel is of type $\varphi \to \psi$ (for example, the type 'if white exchanges knights at d5, she will lose a pawn') is enough to tell us that if $s_1 \models \varphi$ (In s_1 white exchanges knights), $s_2 \models \psi$ (In s_2 white loses a pawn). This is how we classify constraints. The right-to-left part of the biconditional raises more interesting issues. Barwise notes that we capture too much if we take the quantifier as ranging over only actual situations. Then "Humean constant conjunctions of real events" [9] count, when we would rather model only lawlike constraints. How the theory can capture this without buying into modal realism is an interesting question. We will consider this in Section 9.8. □

Now for some more terminological matters.

Definition 9.2

- If $s_1 \overset{c}{\mapsto} s_2$, then s_1 is a *signal* for s_2, and s_2 is a *target* of s_1 relative to channel c.
- The ternary relation $\mapsto$ restricted to c is called the *signalling relation* of c. A pair like $\langle s_1, s_2\rangle$ is called a *signal/target pair* for c.
- If $c \models \varphi \to \psi$ and $s_1 \models \varphi$ but there is no s_2 such that $s_1 \overset{c}{\mapsto} s_2$, then we say that s_1 is a *pseudo-signal* for $c : \varphi \to \psi$.
- If $c \models \varphi \to \psi$ and there are no pseudo-signals for $c : \varphi \to \psi$, then the constraint is *absolute* on c. This is written $c \models \varphi \overset{!}{\to} \psi$. Otherwise the constraint is *conditional* on c.

As an example, suppose that a channel c supports the constraint "if white exchanges knights at d5, she will lose a pawn." Because the channel supports the constraint, its only signal/target pairs where the signal is a knight-exchanging-at-d5 situation must have white-losing-pawn situations as the target. This is merely what it means for the channel to support the constraint. Now consider the possibility where the game ends due to some unforeseen circumstance after the next move, and that this next move was white exchanging knights at d5. Then this move situation (call it s_1) does support the antecedent type, but it must be a *pseudo-signal* for the channel, because there is no next move and no pawn capture. Clearly this constraint is not absolute; when we utter conditionals they rarely classify absolute constraints.

Now we can use the machinery we have so far to begin to model the first of our principles. First some more terminology, to state the principle clearly.

Definition 9.3 Now we can define a notion of information transfer, mediated by an information channel.

- $s_1 : \varphi \xrightarrow{c} s_2 : \psi$ iff $c \models \varphi \to \psi$ and $s_1 \overset{c}{\mapsto} s_2$. This is read: s_1 *being* φ *would carry the information that* s_2 *is* ψ *relative to channel* c.[2]

Now consider the Xerox principle. Why would it be true? Consider the channel c_1 of type $\varphi \to \psi$ and c_2 of type $\psi \to \theta$. We would like there to be a channel c of type $\varphi \to \theta$. This would enable us to prove the Xerox principle in channel-mediated terms. This is what such a channel would have to be like:

Definition 9.4 Channel c is a *sequential composition* of c_1 followed by c_2 just when for all situations s_1 and s_2, $s_1 \overset{c}{\mapsto} s_2$ iff there is an s where $s_1 \overset{c_1}{\mapsto} s$ and $s \overset{c_2}{\mapsto} s_2$.

Given this, it is simple to prove the following:

LEMMA 9.1 *If* c *is a sequential composition of* c_1 *and* c_2 *and* $c_1 \models \varphi \to \psi$ *and* $c_2 \models \psi \to \theta$ *then* $c \models \varphi \to \theta$.

Intuitively, the serial composition of two channels is what you get when you lay the channels end to end. We assume that every two channels compose sequentially. In addition, composition is associative. This is an addition to our definition of an information structure.

Definition 9.5 In each information structure every pair of channels c_1 and c_2 has a unique sequential composition, $c_1; c_2$. In addition, $c_1; (c_2; c_3) = (c_1; c_2); c_3$.

Given this, we can prove the Xerox principle in channel-mediated terms:

LEMMA 9.2 *In any information structure, if* $s_1 : \varphi \xrightarrow{c_1} s_2 : \psi$ *and* $s_2 : \psi \xrightarrow{c_2} s_3 : \theta$ *then* $s_1 : \varphi \xrightarrow{c_1;c_2} s_3 : \theta$.

Proof: A simple matter of writing out definitions and applying the previous lemma. $\triangleleft$

This result explains the validity of the Xerox principle in terms of channels. If $s_1 : \varphi$ carries the information $s_2 : \psi$ (relative to some channel) and $s_2 : \psi$ carries the information $s_3 : \theta$ (relative to a channel), then it is their composition which grounds the information transfer from $s_1 : \varphi$ to $s_3 : \theta$.

To prove the next principle, we need to associate a particular channel to logical validity. The first step is to consider logical validity in terms of situations.

Definition 9.6 Given types φ and ψ, φ *entails* ψ, or *is a subtype of* ψ iff every situation of type φ is also of type ψ. This is written $\varphi \sqsubseteq \psi$.

From here, Barwise identifies a particular "logic channel," which relates every situation to itself, and to no other. In my opinion, this is restrictive. There seems to be no reason to choose between the identity relation and the sub-situation relation (where $s_1 \sqsubseteq s_2$ iff $s_1 \models \varphi \Rightarrow s_2 \models \varphi$ for each φ) to be modelled by the logic channel. We define the logic channel as follows:

Definition 9.7 Each information structure contains a *logic channel*, channel 1. It relates each situation s to all situations that contain it. In other words, $s \overset{1}{\mapsto} s'$ if and only if $s \sqsubseteq s'$.

EXCURSUS: We are diverging from Barwise's account here, primarily because of the treatment of channels in Section 9.3. Barwise takes it that $s \overset{1}{\mapsto} s'$ if and only if $s = s'$. However, the difference is not important. All we need for 1 to work sensibly (in that $1 \models \varphi \to \psi$ if and only if every situation that supports φ also supports ψ) is that $s \overset{1}{\mapsto} s$ and that if $s \overset{1}{\mapsto} s'$ then $s \sqsubseteq s'$. The cases where $\overset{1}{\mapsto}$ is the identity (as in Barwise) and where $\overset{1}{\mapsto}$ is $\sqsubseteq$ (as here) are the two extremes within which the behaviour of 1 can sensibly vary. $\square$

Given this definition we may prove the following:

LEMMA 9.3 **Logic as Information Flow** *Given types* φ *and* ψ, *then* $\varphi \sqsubseteq \psi$ *if and only if* $1 \models \varphi \to \psi$, *if and only if* $1 \models \varphi \overset{!}{\to} \psi$

Proof: By definition, $\varphi \sqsubseteq \psi$ if and only if $1 \models \varphi \to \psi$. We need just show that 1 never has any pseudo-signals for $\varphi \to \psi$. But this is simple. If $s \models \varphi$ then $s \overset{1}{\mapsto} s$ ensures that 1 has no pseudo-signals. $\triangleleft$

It remains to prove our results about conjunction, disjunction and negation. To prove the principle of addition we need a new way of putting channels together. For example, if $s_1 : \varphi \xrightarrow{c} s_2 : \psi$ and $s_1 : \varphi' \xrightarrow{c'} s_2 : \psi'$, what channel $c^?$ would satisfy $s_1 : (\varphi \wedge \varphi') \xrightarrow{c^?} s_2 : (\psi \wedge \psi')$? Clearly some channel would support the type $(\varphi \wedge \varphi') \to (\psi \wedge \psi')$ — it is the channel found by taking c and c' together in parallel. Barwise gives a clear example:

> Television offers a natural example. Here we have two channels, the video and the audio. So let's think of a person watching and listening to the broadcast of a speech. For signals we will take the person's full mental state. For targets, we will take the speech situation. It seems quite natural to think of the former as being linked to the latter by two channels, the audio and the visual. But then it also seems quite natural to put these two together into a third channel, what we could call the *parallel composition* of the two channels [9].

Formally, we have the following:

Definition 9.8 The channel c is a parallel composition of channels c_1 and c_2 provided that for all situations s_1 and s_2, $s_1 \overset{c}{\mapsto} s_2$ iff $s_1 \overset{c_1}{\mapsto} s_2$ and $s_1 \overset{c_2}{\mapsto} s_2$

We expand our definition of information structures appropriately:

Definition 9.9 Every information structure has a total binary operation $\|$ of parallel composition of channels. Parallel composition is commutative, associative, and idempotent, i.e. $c \| c = c$.

Given this, we can prove the principles of addition of information and exhaustive cases.

LEMMA 9.4 *In every information structure the following principles hold:*

- **Addition of Information** *If* $s_1 : \varphi \xrightarrow{c} s_2 : \psi$ *and* $s_1 : \varphi' \xrightarrow{c'} s_2 : \psi'$ *then* $s_1 : (\varphi \wedge \varphi') \xrightarrow{c} s_2 : (\psi \wedge \varphi')$ *where* $c = c_1 \| c_2$.
- **Exhaustive Cases** *Suppose that* $s_1 : \varphi \xrightarrow{c_1} s_2 : (\psi \vee \psi')$, $s_2 : \psi \xrightarrow{c_2} s_3 : \theta$, *and* $s_2 : \psi' \xrightarrow{c_3} s_3 : \theta$. *Then* $s_1 : \varphi \xrightarrow{c} s_3 : \theta$, *where* $c = c_1; (c_2 \| c_3)$.

Proof: Again, a simple case of writing out the definitions. We leave this to the reader. ◁

We ought to give some account of how the two compositions interact. First define a relationship between channels:

Definition 9.10 Channel c_1 is a *refinement* of channel c_2, written $c_1 \preceq c_2$ iff $c_1 = c_1 \| c_2$.

This $c_1 \preceq c_2$ means that using c_1 and c_2 in parallel gives no more information that c_2 alone. So, if c_1 is a refinement of c_2, if $s_1 \overset{c_1}{\mapsto} s_2$ then $s_1 \overset{c_2}{\mapsto} s_2$.

LEMMA 9.5 *If* $c_2 \models \varphi \to \psi$ *and* $c_1 \preceq c_2$ *then* $c_1 \models \varphi \to \psi$, *and* $\preceq$ *partially orders the channels. Furthermore, parallel composition preserves refinement. That is, if* $c_1 \preceq c_2$ *then* $c_1 \| d \preceq c_2 \| d$ *and* $d \| c_1 \preceq d \| c_2$.

Analogously to this we take the corresponding claim in terms of sequential composition as a condition on information structures.

Definition 9.11 In our information structures, sequential composition preserves refinement. That is, if $c_1 \preceq c_2$ then $c_1; d \preceq c_2; d$ and $d; c_1 \preceq d; c_2$.

The last thing we ought to consider is negation. Here Barwise's account is unclear, for he doesn't give a gloss on what it is for a situation to support a negation. The way he wishes to prove the contraposition principle is by way of converses of channels. The channel c^{-1} is a converse of c iff for each s_1 and s_2, $s_1 \overset{c^{-1}}{\longmapsto} s_2$ iff $s_2 \overset{c}{\mapsto} s_1$. The inversion axiom states that each channel has a unique converse, such that $1^{-1} = 1$, $(c_1; c_2)^{-1} = (c_2^{-1}; c_1^{-1})$ and $(c_1 \parallel c_2)^{-1} = c_1^{-1} \parallel c_2^{-1}$.

Barwise claims that it follows from this that if $s_1 : \varphi \xrightarrow{c} s_2 : \psi$, then $s_2 : \sim\varphi \xrightarrow{c^{-1}} s_1 : \sim\varphi$. The only way we could conceive of proving this is as follows: suppose that $s_1 : \varphi \xrightarrow{c} s_2 : \psi$. So $s_1 \overset{c}{\mapsto} s_2$, and $s_2 \overset{c^{-1}}{\longmapsto} s_1$ by the inversion axiom. We want that if $s_2 \models \sim\psi$ then $s_1 \models \sim\varphi$. Suppose that $s_2 \models \sim\psi$. Then we have $s_2 \not\models \psi$ and so, $s_1 \not\models \varphi$. Thus, $s_1 \models \sim\varphi$.

This is satisfactory reasoning, provided we have the assumption that $s \models \sim\varphi$ iff $s \not\models \varphi$. But this is unwarranted if situations are sometimes $\sim$-incomplete or $\sim$-inconsistent, and we have every reason to believe this. (Otherwise they look too much like possible worlds to be comfortable.) Because of this, we must conclude that Barwise's account of negation is in need of reform.

Before going on, we will reiterate the definition of an information structure, with all of the relevant bells and whistles.

Definition 9.12 An *information structure* $\langle S, T, C, \models, \mapsto, \parallel, ; \rangle$ is a structure made up of a set S of *situations*, T of *types*, C of *channels*, a binary relation $\models$ which relates both pairs of situations and types and pairs of channels and constraints and a ternary relation $\mapsto$ relating channels to pairs of situations. This structure must satisfy a number of further conditions:

- Types are closed under binary operations $\wedge$ and $\vee$. Furthermore, for each $s \in S$ and each $\varphi, \psi \in T$, $s \models \varphi \wedge \psi$ if and only if $s \models \varphi$ and $s \models \psi$, and $s \models \varphi \vee \psi$ if and only if $s \models \varphi$ or $s \models \psi$.
- For every $\varphi, \psi \in T$, the object $\varphi \to \psi$ is a *constraint*. The relation $\models$ is extended to channels and constraints in the way indicated: $c \models \varphi \to \psi$ if and only if for each $s_1, s_2 \in S$ where $s_1 \overset{c}{\mapsto} s_2$, if $s_1 \models \varphi$ then $s_2 \models \psi$.
- There is a *logic channel*, channel 1. It relates each situation s to all situations that contain it. In other words, $s \overset{1}{\mapsto} s'$ if and only if $s \sqsubseteq s'$, where $s \sqsubseteq s'$ if and only if whenever $s \models \varphi$, $s' \models \varphi$ too.
- Every pair of channels c_1 and c_2 has a unique *sequential composition* $c_1; c_2$ (such that $s_1 \overset{c_1;c_2}{\longmapsto} s_2$ if and only if there is a situation s such that $s_1 \overset{c_1}{\mapsto} s$ and $s \overset{c_2}{\mapsto} s_2$). In addition, $c_1; (c_2; c_3) = (c_1; c_2); c_3$.
- Every pair of channels c_1 and c_2 has a unique *parallel composition* $c_1 \parallel c_2$ (such that $s_1 \overset{c_1 \parallel c_2}{\longmapsto} s_2$ if and only if $s_1 \overset{c_1}{\mapsto} s_2$ and $s_1 \overset{c_2}{\mapsto} s_2$). In addition, parallel composition is commutative, associative, and idempotent.
- Channel c_1 is a *refinement* of channel c_2, written $c_1 \preceq c_2$ iff $c_1 = c_1 \parallel c_2$.
- Sequential composition preserves refinement. That is, if $c_1 \preceq c_2$ then $c_1; d \preceq c_2; d$ and $d; c_1 \preceq d; c_2$.

In Barwise's initial account of information flow, he shows how a range of model structures in logic, computer science and information theory can each be seen as models of information flow. We'll show that a large class of Routley-Meyer frames also count as models of information flow.

9.4 Frames Model Information Flow

Recall the condition for a channel to support a conditional type.

> The channel $c \models \varphi \to \psi$ if and only if for all situations s_1, s_2, if $s_1 \overset{c}{\mapsto} s_2$ and $s_1 \models \varphi$ then $s_2 \models \psi$.

Clearly this is reminiscent of the modelling condition of conditionals in frame semantics. If we take it that channels *are* situations, then the condition *is* that of the conditional in the frame semantics, where $\mapsto$ is R.

In frame semantics $x \overset{y}{\mapsto} z$ means that the conditional information given by y applied to x results in no more than z. This results in the plausible *monotonicity condition*

$$\text{If } x' \sqsubseteq x,\ y' \sqsubseteq y \text{ and } z \sqsubseteq z' \text{ and } x \overset{y}{\mapsto} z \text{ then } x' \overset{y'}{\mapsto} z'.$$

It is natural to take the serial composition $x;y$ to be contained in situation z just when $x \overset{y}{\mapsto} z$. This is because $x \overset{y}{\mapsto} z$ is read as "applying y to x keeps you in z." But serially composing x and y is just applying the information from y to that in x in order to get a new channel. So, if the application of y to x is bounded above by z, we must have $x;y$ contained in z. And *vice versa*. So, from now we will read $x \overset{y}{\mapsto} z$ as $x;y \sqsubseteq z$ and vice versa.

What would associativity of channels mean in this context? We simply require that $(x;y);z \sqsubseteq u$ if and only if $x;(y;z) \sqsubseteq u$ for each u. But this comes out as follows. $(x;y);z \sqsubseteq u$ if and only if for some v, $x;y \sqsubseteq v$ and $v;z \sqsubseteq u$. In other words, for some v, $x \overset{y}{\mapsto} v$ and $v \overset{z}{\mapsto} u$. Conversely, $x;(y;z) \sqsubseteq u$ if and only if for some w, $x;w \sqsubseteq u$ and $y;z \sqsubseteq w$, which can be rephrased as $x \overset{w}{\mapsto} u$ and $y \overset{z}{\mapsto} w$. Given our rewriting of sequential channel composition in terms of the channel relation $\mapsto$ we have an associativity condition in terms of $\mapsto$ alone.

Some situations are special, in that they contain *logic*. That is, these situations act as channels from situations to themselves. Let I be the class of these situations. Clearly then, if for some $z \in I$, we have $x \overset{z}{\mapsto} y$, then $x \sqsubseteq y$. Also, for each situation x, there is a $z \in I$ where $x \overset{z}{\mapsto} x$. (To get $\mathbf{LI}^+$ we would also take dual versions of these — that if for some $z \in I$, $z \overset{x}{\mapsto} y$ then $x \sqsubseteq y$ and for each x there is a $z \in I$ where $z \overset{x}{\mapsto} x$, in order to model the backwards $\leftarrow$ of $\mathbf{LI}^+$.)

This will be enough to start our definition of a frame modelling information flow.

Definition 9.13 A *bare frame* is a quadruple $\langle S, \mapsto, I, \sqsubseteq \rangle$, where S is a set of *situations*, $\mapsto$ is a ternary relation on S, I is a subset of S, the *logic cone* and $\sqsubseteq$ is a partial order on S. The objects satisfy the following further conditions.

- If $x' \sqsubseteq x$, $y' \sqsubseteq y$ and $z \sqsubseteq z'$ and $x \overset{y}{\mapsto} z$ then $x' \overset{y'}{\mapsto} z'$
- If $z \in I$ and $z \sqsubseteq z'$, then $z' \in I$.

- If for some $z \in I$, $x \overset{z}{\mapsto} y$, then $x \sqsubseteq y$.
- For each x, there is a $z \in I$ where $x \overset{z}{\mapsto} x$.
- $(\exists v)(x \overset{y}{\mapsto} v$ and $v \overset{z}{\mapsto} u)$ if and only if $(\exists w)(x \overset{w}{\mapsto} u$ and $y \overset{z}{\mapsto} w)$.[3]

Now that we have the structures defined, we need to see that these structures really model the axioms, by defining parallel and serial composition.

Take situations a and b. Their serial composition ought to be the 'smallest' situation x (under $\sqsubseteq$) such that $a \overset{b}{\mapsto} x$ given our motivation of identifying $a \overset{b}{\mapsto} c$ with $a; b \sqsubseteq c$. However, nothing assures us that such a minimal situation exists. For there may be two candidate situations which agree with regard to all conditionals, but disagree with regard to a disjunction $p \vee q$. As situations are prime, neither of these is minimal. Instead of requiring that such a situation exist, we will model the serial composition of these two situations as the *set* $\{x : a \overset{b}{\mapsto} x\}$. If we take a set to support the type A just when each of its elements supports A, the set $\{x : a \overset{b}{\mapsto} x\}$ will work as the serial composition of a and b. It may be considered to be a 'non-prime situation,' or merely as the information shared by a collection of situations. From now on we take our channels to be sets of situations like this. A channel can be taken to be part of a situation just when the situation is an element of the channel. Let's make things formal with a few definitions.

Definition 9.14 Given a bare frame $\langle S, \mapsto, I, \sqsubseteq \rangle$

- $X \subseteq S$ is a *cone* iff for each $x \in X$, if $x \sqsubseteq y$ then $y \in X$. (So I is a cone, the logic cone).
- If X is a cone, $X \models A$ iff $x \models A$ for each $x \in X$.
- If X, Y and Z are cones, $X \overset{Y}{\mapsto} Z$ if and only if for every $z \in Z$ there are $x \in X$ and $y \in Y$ where $x \overset{y}{\mapsto} z$.
- If X and Y are cones, $X \sqsubseteq Y$ if and only if $Y \subseteq X$. In addition, $X; Y = \{z : X \overset{Y}{\mapsto} z\}$, $X \parallel Y = \{z : X \sqsubseteq z \text{ and } Y \sqsubseteq z\}$.
- Given an evaluation $\models$ on our frame, $X \models A$ iff for each $x \in X$, $x \models A$.
- For each situation x, $\uparrow x$ is the principal cone on x. In other words, $\uparrow x = \{x' : x \sqsubseteq x'\}$.

Given these definitions, it is not difficult to prove the following results.

LEMMA 9.6 *Given a bare frame* $\langle S, \mapsto, I, \sqsubseteq \rangle$ *with an evaluation* $\models$

- $X \models A \to B$ *iff for each pair of cones* Y, Z, *where* $Y \overset{X}{\mapsto} Z$, *if* $Y \models A$ *then* $Z \models B$.
- $X \models A \to B$ *iff for each pair of situations* y, z, *where* $\uparrow y \overset{X}{\mapsto} \uparrow z$, *if* $y \models A$ *then* $z \models B$.
- $X \models A \wedge B$ *iff* $X \models A$ *and* $X \models B$.
- $X \models A \vee B$ *iff for each* $x \in X$, *either* $x \models A$ *or* $x \models B$.
- $\uparrow x \sqsubseteq Y$ *iff for each* $y \in Y$, $x \sqsubseteq y$.
- $X \sqsubseteq \uparrow y$ *iff* $y \in X$.
- $(\exists v)(X \overset{Y}{\mapsto} v$ *and* $v \overset{Z}{\mapsto} U)$ *iff* $(\exists w)(X \overset{w}{\mapsto} U$ *and* $Y \overset{Z}{\mapsto} w)$.
- $\uparrow x \sqsubseteq \uparrow y$ *iff* $x \sqsubseteq y$.
- $\uparrow x \overset{\uparrow y}{\mapsto} \uparrow z$ *iff* $x \overset{y}{\mapsto} z$.
- $\uparrow x \models A$ *if and only if* $x \models A$.

Proof: Straight from the definitions. We leave them as an exercise. ◁

Because of the last three results in that lemma, principal cones will do for situations whenever they occur. From now, we will slip between a principal cone and its situation without mentioning it.

The significant result is that $X;Y$ really is the serial composition of X and Y. In other words, we can prove the following:

LEMMA 9.7 *For all cones* X *and* Y, *and for all situations* a *and* c, $a \xmapsto{X;Y} c$ *iff there is a situation* b *such that* $a \xmapsto{X} b$ *and* $b \xmapsto{Y} c$.

Proof: Suppose that $a \xmapsto{X;Y} c$. Then for some $d \in X;Y$, $a \xmapsto{d} c$. However, if $d \in X;Y$ we must have an $x \in X$ and a $y \in Y$ where $x \xmapsto{y} d$. So, $x \xmapsto{y} d$ and $a \xmapsto{d} c$. This means that for some b, $a \xmapsto{x} b$ and $b \xmapsto{y} c$ by one half of the associativity condition. This means that $a \xmapsto{X} b$ and $b \xmapsto{Y} c$ as desired.

The converse merely runs this proof backwards and we leave it as an exercise. ◁

Then serial composition is associative because of our transitivity condition on modelling conditions.

LEMMA 9.8 *In any bare frame, for any cones* X, Y *and* Z, $X;(Y;Z) = (X;Y);Z$.

Proof: If $w \in X;(Y;Z)$ then $X \xmapsto{Y;Z} w$, which means that for some $x \in X$ and $v \in Y;Z$, $x \xmapsto{v} w$. Similarly, $v \in Y;Z$ means that for some $y \in Y$ and $z \in Z$, $y \xmapsto{z} v$. But this means that for some u, $x \xmapsto{y} u$ and $u \xmapsto{z} w$, which gives $u \in X;Y$ and so, $w \in (X;Y);Z$ as desired. The converse proof is the proof run backwards, as usual. ◁

The logic cone I then satisfies the condition for the logic channel.

LEMMA 9.9 *In any bare frame,* $x \xmapsto{I} y$ *if and only if* $x \sqsubseteq y$.

Proof: Trivial. If $x \xmapsto{I} y$ then $x \xmapsto{z} y$ for some $z \in I$, which merely means that $x \sqsubseteq y$. Conversely, there is some $z \in I$ where $x \xmapsto{z} x$, which gives $x \xmapsto{z} y$ by monotonicity, and so, $x \xmapsto{I} z$ as desired. ◁

As things stand, parallel composition may not work as we intend. We may have cones X and Y for which the parallel composition is empty (that is, there is no z where $X \sqsubseteq z$ and $Y \sqsubseteq z$), but this means that $a \xmapsto{X\|Y} b$ does not hold (look at the clauses). However, we may still have $a \xmapsto{X} b$ and $a \xmapsto{Y} b$. We need an extra modelling condition.

Definition 9.15 A bare frame $\langle S, \mapsto, I, \sqsubseteq\rangle$ is an *information frame* if and only if for each a, b, x, y where $a \xmapsto{x} b$ and $a \xmapsto{y} b$, there is a z where $x \sqsubseteq z$ and $y \sqsubseteq z$ and $a \xmapsto{z} b$. Parallel composition in an information frame is defined as you would expect. $X \| Y = \{z : X \sqsubseteq z \text{ and } Y \sqsubseteq z\}$.

LEMMA 9.10 *In any information frame, for all cones* X *and* Y, *and for all situations* a *and* b, $a \xmapsto{X} b$ *and* $a \xmapsto{Y} b$ *iff* $a \xmapsto{X\|Y} b$. *Furthermore, parallel composition then does what we would expect of it:* $X \| X = X$, $X \| Y = Y \| X$ *and* $X \|(Y \| Z) = (X \| Y) \| Z$. *In fact* $X \| Y = X \cap Y$.

Proof: Trivial. Clearly, if $a \overset{X \parallel Y}{\longmapsto} b$ then $a \overset{X}{\mapsto} b$ and $a \overset{Y}{\mapsto} b$ by monotonicity. The converse holds by the definition of an information frame. The properties of information frames follow from the fact that $\sqsubseteq$ is a partial order. The fact that parallel composition turns out to be intersection follows from the fact that it is defined on cones on the partial order. ◁

The relation $\preceq$ of refinement is simply $\sqsupseteq$; $X \preceq Y$ iff $X \parallel Y = X$, iff $X \cap Y = X$ iff $X \subseteq Y$ iff $X \sqsupseteq Y$.

The last thing to show is that serial composition preserves refinement. We need to show the following:

LEMMA 9.11 *In any information frame, if* $X_1 \sqsubseteq X_2$ *and* $Y_1 \sqsubseteq Y_2$ *then* $X_1; Y_1 \sqsubseteq X_2; Y_2$.

Proof: This follows immediately from the hereditary conditions on $\mapsto$. ◁

So, our frame structures are models of information flow. The corresponding logic is a weak one. We don't need to assume contraction, weakening or permutation. The only conditions above **DW** are prefixing and its converse. So, we have shown that the semantics underlying a wide class of our logics have a home within Barwise's account of information flow.

9.5 Frames Are the Best Model of Information Flow

As we said before, Barwise's original paper [9] shows that model structures like those of classical logic and intuitionistic logic also model information flow. If we were only able to put our logics into this class, this would not say very much for our logics or for the cause of contraction-freedom. Why favour systems without contraction? Why favour the distinctions that frame semantics can draw (such as contraction failing) and the identifications it makes (such as identifying channels with situations)?

Why Frames? Firstly, frames are the sensible model of information flow, because they remain faithful to the original intuitions about the meanings of conditional utterances. If we say that situations and channels are totally distinct, then we have a problem. We decided that a conditional statement "if S_1 then S_2" has the constraint $\varphi \to \psi$ as descriptive content, and a particular *channel* c as demonstrative content. But we already decided that other statements have *situations* as their demonstrative content. Why are conditional statements any different? To avoid such an arbitrary distinction, we must admit a relationship between situations and channels. If the conditional "if S_1 then S_2" has a situation s as its demonstrative content, then we may take s itself to be a channel between situations. This is admitted in the frame semantics. Each situation is a channel and arbitrary *cones* of situations are also channels — chiefly to deal with composition.

EXCURSUS: If you are prepared to take composition as a *relation* between situations and not as a function, we could abandon cones completely. We could just say z is *a* serial composition of x and y if $x \overset{y}{\mapsto} z$ (i.e. $x; y \sqsubseteq z$) and z is *a* parallel composition of x and y if $x \sqsubseteq z$ and $y \sqsubseteq z$. Then, given axioms such as: each pair of situations has both a serial and a parallel composition, all of Barwise's principles of information flow will still hold, without requiring recourse to cones or explicit operations on situations. □

In other words, once we recall a major motivating application of information flow (modelling conditionals in situation semantics in terms of regularities grounded in the world) the frame semantics identification of channels with situations is the natural conclusion.

Why No Contraction? We know that frames model information flow. It doesn't follow that contraction-freedom is thereby justified. Even though Barwise specified that serial composition be associative, leaving open whether it was symmetric, idempotent or whatever, this only leaves the way open for contraction-freedom, it doesn't motivate it.

To motivate contraction-freedom we need to find a counterexample to $x; x \sqsubseteq x$. But this is simple. For many channels, you may eke out more information by repeated application. For a number of examples, consider the information theoretic aspects of the applications in the previous chapters. The conditionals defined in the context of vagueness are an example. Take the degrees of stretch to be situations. Then a channel may support a conditional and its antecedent without supporting its consequent, as we reasoned before. But there are many more reasons to reject contraction. We need only find a domain in which using a channel twice (serially) yields more information than using it once. We'll just sketch one such application, and leave the reader to think of more.

Take situations to be mathematical proofs, and we will take the information content to be the *explicit content* of the proof.[4] That is, the things that are stated in the proof. We can model the information flow from a proof to another proof (which may contain the first proof as a part) by way of information links that relate proof-situations by means of the deductive principles of mathematics. For example, one such rule dictates that

$$\text{If } n \text{ is even, so is } n+2.$$

This may be warranted by a channel x, so that if y and z are proof situations where $y \overset{x}{\mapsto} z$, and y supports the claim '6 is even,' then z supports the claim '8 is even.' The proof z may have been produced from the proof y by applying the rule of inference we've seen, and the channel x indicates (or warrants) that application.

Now a proof may have '8 is even' as a part of its explicit content without '10 is even' also being a part of its content. To get that from the initial proof situation y, we need to apply the rule *twice*, and so, use the channel x twice. The channel $x; x$ therefore warrants more than x. We may have $y \overset{x}{\mapsto} z$ without $y \overset{x;x}{\longmapsto} z$.

9.6 Interpreting Conditional Sentences

Now that we have the formalism in place, we can describe the relationship between channels (or situations) and conditionals. This will also give the wherewithal to give an account of some seemingly paradoxical arguments using conditionals.

In what we have seen, all declarative utterances have a demonstrative content (a situation) and a descriptive content (a type). This is no different for conditional utterances, which have a situation as demonstrative content, and a type of the form $\varphi \to \psi$ as descriptive content. The truth or otherwise of the conditional depends on whether the situation described is of the type or not. This situation relativity gives us the means to give an account of odd-sounding arguments.

The first is a putative counterexample to the Xerox principle. Consider the two conditionals:

> If an election is held on December the 25th, it will be held in December.
>
> If an election is held in December, it will not be held on December the 25th.

By a naïve application of Xerox principle we could argue from these two claims that if an election is held on December the 25th, it will not be held on December the 25th. This is an odd conclusion to draw. The analysis in terms of channels and constraints can help explain the puzzle without rejecting the Xerox principle.

Firstly, consider the situation described by the first conditional. This situation — in its job as a channel — pairs election-on-December-the-25th situations with election-in-December situations. Presumably this channel arises by means of logical truth, or our conventions regarding dates and months. Quite probably, it pairs each election situation with itself. If the antecedent situation has the election on December the 25th, then the consequent situation (the same one) has the election in December. There is little odd with this scenario.

The second situation is different. It pairs antecedent election-in-December situations with consequent election-not-on-December-the-25th situations. Given the plausible assumption that it pairs antecedent situations only with identical consequent situations (or at least, consequent situations not incompatible with the antecedent situations — so it will not pair an antecedent situation with a consequent situation in which the election occurs at a *different* date) it will 'filter out' antecedent situations in which the election is held on Christmas Day. In our parlance, these situations are pseudo-signals for the constraint. These aberrant situations are not related (by our channel) with any other situation at all. The channel only relates 'reasonably likely' election situations with themselves, and so, it supports the constraint that elections in December are not held on Christmas Day just because it doesn't relate those (unlikely) situations in which an election is held on that day.

Given these two channels, it is clear that their composition supports the constraint 'if there is an election on December the 25th, then the election is not on December the 25th' simply because there are no signal/target pairs for that channel in which the signal is a situation in which the election is on December the 25th. The composition of the channels filters out these antecedent situations by construction. That channel supports other odd constraints like 'if there is an election on December the 25th, then Queensland has won the Sheffield Shield for the last twenty years.' This does not tell us anything enlightening about what would happen were an election to actually occur on December the 25th — it only tells us that this particular channel has ruled that possibility out.

Composition of channels may 'filter out' possibilities (like elections held on Christmas day) that will later become relevant. Then we typically broaden the channels to admit more possibilities. (This is akin to expanding the set of 'nearby possible worlds' used to evaluate counterfactuals on the Lewis-Stalnaker accounts.) Typically when we utter a counterfactual conditional we mean that situations classified by the antecedent will feature in signal/target pairs of the channel being utilised. (Otherwise, why utter the conditional?) In cases like this argument, the composed channel is not like this. The antecedent situations being described do not feature as in signal-target pairs of the channel being classified. So, the conditional given by the Xerox principle is not the same as the conditional you would typically be expressing had you said

If an election is held on December the 25th, it will not be held on December the 25th.

Had you said that (and it is a strange thing to say) then most likely, the channel being classified would have as signal/target pairs some situations in which the election is held on December the 25th. (Otherwise, why call attention to their possibility?) And if this is so, the conditional you express by asserting the sentence will differ from that arising from the Xerox principle but this only points out the channel relativity of conditionals. This is parallel to the situation-theoretic fact that propositions are situation relative. So, the principle itself is sound, but difficulties like these must keep us alert to the way it is used.

As was the case with the Xerox principle, we can use the channel-theoretic account to explain the oddity of certain 'deductions' using conjunction. For example, given a number of background conditions

If it snows, I won't be surprised.

If it hails, I won't be surprised.

could be both true. It may be quite cold, and both snow and hail could be possible given the current weather conditions. Furthermore, my expectations are attuned to the relevant facts about the weather, and so, I won't be surprised under either of those conditions. However, in this situation, there is no guarantee that

If it both snows and hails, I won't be surprised.

because the combination of snow and hail is a very rare thing indeed — and I may be aware of this. This appears to be a counterexample to the principle of addition of information (collapsing the two conjuncts in the consequent to one). Yet, as with the Xerox principle, this is not a real counterexample. What's more, the account in terms of channels can help us explain the surprising nature of the 'deduction.'

Consider the channels supporting the original two claims. Obviously they do not relate all snowing or hailing signal-target pairs with consequent mental states, because for some snowing or hailing situations (ones that are combined) I *am* surprised. So, the channels supporting these claims must ignore the possible (but rare) situations in which snow and hail coincides. In other words, snow-and-hail situations are pseudo-signals for this constraint. This is understandable, because it is a rare situation to encounter. Now when we consider the parallel composition of the two channels, it is simple to see that it has no signal-target pairs where it is snowing and hailing in the signal situation. In each of the original channels, these possibilities were filtered out, so they cannot re-emerge in their parallel composition. The third conditional is supported by the parallel composition of channels only vacuously. The composed channel does not relate any snowy-and-haily situations.

Were we to say 'if it both snows and hails, I won't be surprised' the channel classified would (usually) not be one that filters out odd snowy-and-haily situations, because we have explicitly mentioned that situation as a possibility. Again, we must be careful to not identify the conclusion of the addition of information principle with a claim we may express ourselves. For each declarative utterance, there is a corresponding situation or channel that is classified. Different utterances could well classify different situations or channels.

In this way we can use the formalism to explain the oddity in certain conditional argument forms. They are sensitive to the situations being described, which can vary from premise to conclusion, without this fact being explicit.

9.7 Negation

As we've commented before, Barwise's account of negation has to be wrong, if we are concerned to model a useful negation (for which situations could be incomplete or inconsistent). Boolean negation isn't the kind of negation that you would want to use as a situation-theoretic analysis of our natural language 'not.' Rather more plausible is to say that situations can both support and undermine situation-types. We have the clauses we've seen in Chapter 6 for conjunction, disjunction and negation:

- $x \models A \wedge B$ iff $x \models A$ and $x \models B$,
- $x \dashv A \wedge B$ iff $x \dashv A$ or $x \dashv B$,
- $x \models A \vee B$ iff $x \models A$ or $x \models B$,
- $x \dashv A \vee B$ iff $x \dashv A$ and $x \dashv B$,
- $x \models {\sim}A$ iff $x \dashv A$,
- $x \dashv {\sim}A$ iff $x \models A$.

For the conditional we use the standard positive condition:

- $x \models A \to B$ iff for all $y, z \in W$ if $y \overset{x}{\mapsto} z$ and $y \models A$ then $z \models B$.

As a result, contraposition will fail, but we have no qualms about that, given the examples we have seen.

The issue to address is the negative condition for a conditional. We may think that the plausible condition is:

- $x \dashv A \to B$ iff there are y, z where $y \overset{x}{\mapsto} z$, $y \models A$ and $z \dashv B$.

This is because we may think that a situation undermines a conditional just when there is a concrete counterexample to it. This is one possibility, but it conflicts with plausible intuitions about containment relations among situations. It is desirable that if $x \dashv A \to B$ and $x \sqsubseteq x'$ then $x' \dashv A \to B$ too. However, this is not guaranteed under the condition given. The reason is as follows: we may have $x \dashv A \to B$ by way of a counterexample pair $\langle y, z\rangle$ where $y \overset{x}{\mapsto} z$, $y \models A$ and $z \dashv B$. However, as we expand the situation out to a larger $x' \sqsupseteq x$, x' may no longer act as a channel from y to z, so the pair $\langle y, z\rangle$ will no longer act as a counterexample. Situations which relate many pairs of situations by and large have few true conditionals and *vice versa*, because to get a true conditional you need to check *every* pair that are related by the appropriate situation. There will be more opportunities to get a pair of situations related by the original situation, such that the antecedent of the conditional is supported by the first, and the consequent *not* supported by the second. But then all the same, there are more opportunities to *falsify* conditionals in just the same kinds of situations. So, on this suggestion, situations that are weak on true conditionals tend to be strong on falsifying conditionals, and *vice versa*. This seems strange. It is odd to say that a pair $\langle y, z\rangle$ where $y \overset{x}{\mapsto} z$, provides a counterexample to $\varphi \to \psi$ at x on its own. It is quite possible that when we fill x out to a larger situation x', the counterexample disappears because the expanded channel may not relate y to z. This motivates a different condition for undermining conditionals:

- $x \dashv A \to B$ iff for each $x' \sqsupseteq x$ there are y, z where $y \overset{x'}{\mapsto} z$, $y \models A$ and $z \dashv B$.

Now it is pretty clear that if $x \models A$ and $x \sqsubseteq y$ then $y \models A$ (and similarly for $\dashv$). We retain monotonicity.

The resulting logic will differ from standard ones we have considered. We will retain double negation laws, and de Morgan laws, but all forms of contraposition are lost. Given the counterexamples to contraposition that apply in many fields of information flow, this is not a loss. However, I don't have an axiomatisation of the logic that results. This is an open problem.

9.8 Issues to Ponder

There are a number of important issues to consider, the answers to which will be important for the resulting logic.

- What *are* situations?

The obvious answer is "bits of the world," and for many applications this may be sufficient. However, for conditionals, it isn't. Consider false conditionals with false antecedents. To make

If Queensland win the Sheffield Shield, grass will be blue

false at a situation x we need to have a situation pair $\langle y, z\rangle$ where $y \overset{x}{\mapsto} z$ where $y \models$ Queensland win the Sheffield Shield and $z \dashv$ grass will be blue. This requires the existence of a situation y that supports the claim that Queensland win the Sheffield Shield. As we all know, this is patently false, so no actual situation supports it. It follows that the conditional is true (which is an unsavoury conclusion) or there are some non-actual situations.

It follows that a decent account of conditionals in this formalisation must involve non-actual situations — so these don't qualify as bits of the world (in the same way that actual situations do, anyway). Barwise agrees. His answer to the question is that situations ought to be seen as

> mathematical objects for modelling possibilities, and not as real but not actual situations. [9]

But this won't suffice as an explanation — at least if this semantics is to work as an account of anything. Because if a *model* is to have any explanatory force, the things in the model must correspond to *something* real. If we have a mathematical model for a possibility, then if it does any explanatory work (which it does) there must be something *real* that corresponds to it, and that grounds the explanation in the Way Things Are. If these things are not "real but not actual situations" it would be interesting to hear what they are. Giving an account of these that doesn't amount to realism about (particular) non-actual situations is exactly parallel to the task of giving a non-realist account of possible worlds. Calling them "mathematical models" is honest, but it only pushes the question back to those who want to know what the model actually *models*.

For this approach to have any chance of working without committing us to modal realism, you must explicate the notion of "modelling possibilities." At face value this does seem to involve a relationship between models and the possibilities they purport to model. However, there may be another way to cash out the conception: we can say that x models a possibility (or *represents* a possibility) if x *would* model (or represent) a situation, were things different in some relevant respect. To make this work we need to spell out what way things are allowed to vary, and be more specific about the representation relation. However, it is plausible that some explanation like this might work. Chris Menzel has done this for possible worlds semantics [83], and a broader account, giving an analysis of conditionals in terms of situations, might also work.

Taking this line would result in the analysis being circular in one sense. Cashing out the notion of representation requires using conditionality or possibility — so we will not be giving a reductive account of conditionals. On this semantics, possibility or conditionality will be primitives. However, they will be primitives that are closely associated with other concepts such as the channeling relation between situations, and this, as in all formal semantics, will give us a helpful regimentation of our intuitions about conditionals, and it will give us a new way to analyse their semantic content.

- Are there inconsistent situations? Are there *actual* inconsistent situations?

Clearly for modelling the doxastic states of agents in terms of situations, inconsistent situations are important. Are there others? Are any of these situations actual? (This is the question of true contradictions.) To answer this swiftly would be to beg too many important questions. So we will flag the issue here, and not come back to it.

- Are there worlds?

Do we take all situations to be small, or are there some which correspond to possible worlds? If there are worlds, then we will have a good grounding of world validity in situation semantics. We can explain world validity, as truth preservation across all possible worlds without sacrificing the logic we already have. We can go further and posit that worlds are complete — excluded middle holds for them. This doesn't mean that excluded middle holds in all situations in the logic cone. Whether assuming excluded middle for worlds (and not 'logic') results in a different logic than otherwise is a question for further investigation. Suffice to say here that this conception of the relationship between logic and truth gives more grounding to Slaney's comment in his 'Metacompleteness' paper [143] in which he comments that though excluded middle may be *true* (and true of necessity) this does not mean that it ought to be accounted as a theorem of logic. Logic is about valid inference, and excluded middle does not record an inference. On our picture, excluded middle may be true of worlds, without being among the claims supported by the logic channel.

- What grounds the channelling relation?

This is a very hard question. To answer it well requires a considered view of the metaphysics of modality and conditionality. Let me only sketch the possibilities here. There seem to be at least three. One is to be a realist about non-actual situations (as Lewis is) and to give an account of a natural relationship between these situations, which is a candidate for the channelling relationship. This is going to be hard, because in many cases, the channelling relationship seems to be grounded in our conventions. (How else might a knocking sound indicate that someone is at the door?) A second approach is to still be a realist about non-actual situations, but consider the channelling relationship as one that arises out of the truths of conditionals at situations. This seems to be similar to Lewis' account of counterfactual conditionals, and we have already discussed it. Thirdly, you could abandon realism about non-actual situations, and take the channelling relationship (now in a model, since there are not enough real situations out there to relate to one another) to again arise out of the truths of conditionals in situations. This approach, as the one before it, will not be reductive, but it will be a structural account of the truth conditions for conditionals in situations. There are clearly more issues to discuss here, but they must be left for another occasion.

- Are all worlds consistent? Or are they complete?

It is standard to assume that the world is consistent and complete (and that this is a requirement for *all* possible worlds). There are alternative analyses of the paradoxes that still admit worlds, but deny that they are consistent and complete. These possibilities are clearly compatible with situation semantics as we've described it. So, our structure provides a home for both the classical approach of Barwise and Etchemendy and the non-classical approaches we've considered in earlier chapters. Further work will have to be done to see what differences there are between the two accounts in terms of modelling conditionals.

- How do we model quantification?

The only 'official' modelling of quantification in ternary frames is quite baroque [43]. Perhaps the alternative modelling in terms of situation semantics can do better. In situation semantics, it is possible to model quantified claims at a situation in terms of relations that hold between types (or universals) at that situation. An interesting question for further discussion is how this works out in practice. What inferences that involve quantification and the conditional are actually *valid*?

9.9 Notes

This chapter was presented to the Automated Reasoning Project at the Australian National University in April 1993. I'm grateful for comments and criticism from those

present. Especially Frank Jackson and Peter Menzies who asked me about elections in December, and the coincidence of snow and hail.

[1]Clearly, you could introduce a negation-like operator '$-$' and require that $s \models -\varphi$ iff $s \not\models \varphi$, but then you have to put up with things like almost every situation will support both

$$-(\textsf{Queensland won the Sheffield Shield}) \text{ and}$$

$$-(\textsf{Queensland has not won the Sheffield Shield})$$

by virtue of not containing any (positive) information about the Sheffield Shield at all. But this is just *boolean negation* again. If you're happy with such an account, and the fact that you lose the *hereditary* nature of situations (there is no case where a situation supports a proper subset of the things supported by another) then you're free to use boolean negation: if you can find a use for it.

[2]Barwise defines another notion of information flow, by setting: $s_1 : \varphi \overset{c}{\Longrightarrow} s_2 : \psi$ iff $s_1 : \varphi \xrightarrow{c} s_2 : \psi$ and $s_1 \models \varphi$. This is read: s_1 *being* φ *does carry the information that* s_2 *is* ψ *relative to channel* c. This notion does little work of its own, and we will not mention it again.

[3]These are clauses for *unsimplified* ternary relational semantics. This seems necessary because of the connection between $\mapsto$, $\sqsubseteq$ and the logic channel. It would be unmotivated in what follows to assume that there is a situation g such that $x \overset{g}{\mapsto} y$ if and only if $x = y$, because of the monotonicity condition.

[4]And its real logical consequences, as defined in the situation-theoretic account. So, if $A \wedge B$ is a part of the information content of a proof, so are A, B and $A \vee C$.

Chapter 10
The Lambek Calculus Extended

I cannot yet reduce my Observations
to a *calculus*.

— Monsieur HEVELIUS from DANTZICK
Letter in the *Philosophical Transactions of the Royal Society of London* [167]

10.1 Introduction

Contraction-free logics appear in an application to theoretical linguistics. Consider a sentence. It is made up of a number of words, each of which have different *types*. Some are nouns, others are adjectives others are verbs, and so on. In addition, different parts of sentences have different types. For example, some are noun phrases, others are adverbial phrases, and so on. It is possible to mathematically analyse these parts of speech by means of a formal system. For example, given a basic categorisation of some words:

Kim, Whitney	NP
documentary, cartoon	N

it is possible to give an account of *other* words in terms of what parts of speech they result in when combined with these. For example: jumps is of type NP $\rightarrow$ S, because when combined with a part of speech of type NP on the left, it results in something of type S (a sentence). So, Kim jumps, and Whitney jumps are sentences. The word jumps is *not* of type S $\leftarrow$ NP, as jumps Kim is not a sentence. So jump, when combined with something of type NP on the right, does not yield a sentence. Instead, jumps Kim is of type $(\mathsf{NP} \rightarrow \mathsf{S}) \circ \mathsf{NP}$. That is, it is something of type $(\mathsf{NP} \rightarrow \mathsf{S})$ juxtaposed to the left of something of type NP. So, there are three binary operations on types: $\circ$, $\rightarrow$ and $\leftarrow$. The *Lambek Associative Calculus*[1] is given by a Gentzen calculus with antecedents of rules bunched in lists. Then $x_1, x_2, \ldots, x_n \rhd y$ is read as "the juxtaposition of something of type x_1 with something of type x_2 with ... something of type x_n is also of type y."

$$x \rhd x \qquad \frac{X \rhd x \quad Y \rhd y}{X, Y \rhd x \circ y} \qquad \frac{X, x, y, Y \rhd z}{X, x \circ y, Y \rhd z}$$

$$\frac{X \rhd x \quad Y, y, Z \rhd z}{Y, X, x \rightarrow y, Z \rhd z} \quad \frac{x, X \rhd y}{X \rhd x \rightarrow y}$$

$$\frac{X \rhd x \quad Y, y, Z \rhd z}{Y, y \leftarrow x, X, Z \rhd z} \quad \frac{X, x \rhd y}{X \rhd y \leftarrow x}$$

The standard cut-elimination proof applies, to show that cut is admissible in this system. This system is $\mathbf{L}_{\rightarrow\circ\leftarrow}$, given the conservative extension result from Chapter 7 (provided that you note that ';' has been exchanged for ','.)

Given this type calculus, it is possible to give an account of the way different parts of a sentence fit together. For example, here is a type analysis of a sentence:

$$\dfrac{\dfrac{\dfrac{\overset{\text{Kim}}{\mathsf{NP}}\quad \overset{\text{showed}}{\mathsf{NP} \to ((\mathsf{S}\leftarrow \mathsf{NP})\leftarrow \mathsf{NP})}}{(\mathsf{S}\leftarrow \mathsf{NP})\leftarrow \mathsf{NP}}\quad \overset{\text{Whitney}}{\mathsf{NP}}}{\mathsf{S}\leftarrow \mathsf{NP}}\quad \dfrac{\overset{\text{a}}{\mathsf{NP}\leftarrow \mathsf{N}}\quad \overset{\text{documentary}}{\mathsf{N}}}{\mathsf{NP}}}{\mathsf{S}}$$

A step in the analysis of the form:

$$\frac{A \quad B}{C}$$

means that the part of the sentence (say X) above A is of type A, and the part of the sentence (say Y) above B is of type B, and that as a result of this, the part XY is of type C.

Any complex sentence has more than one analysis, corresponding to different 'bracketings' of the sentence. For example:

$$\dfrac{\dfrac{\dfrac{\overset{\text{Kim}}{\mathsf{NP}}\quad \overset{\text{showed}}{\mathsf{NP} \to ((\mathsf{S}\leftarrow \mathsf{NP})\leftarrow \mathsf{NP})}}{(\mathsf{S}\leftarrow \mathsf{NP})\leftarrow \mathsf{NP}}}{\mathsf{S}\leftarrow \mathsf{NP}\circ \mathsf{NP}}\quad \dfrac{\overset{\text{Whitney}}{\mathsf{NP}}\quad \dfrac{\overset{\text{a}}{\mathsf{NP}\leftarrow \mathsf{N}}\quad \overset{\text{documentary}}{\mathsf{N}}}{\mathsf{NP}}}{\mathsf{NP}\circ \mathsf{NP}}}{\mathsf{S}}$$

This analysis shows that the fragment Whitney a documentary has type $\mathsf{NP} \circ \mathsf{NP}$, so it is the sort of thing that can be concatenated to the right of Kim showed to get a sentence.

Under this interpretation, the failure of contraction is immediately obvious. A verb concatenated with a verb is rarely another verb. So, this account gives us another use for contraction-free logics.

We can show that the formalism fits the interpretation nicely by formalising the notion of a typing scheme for a language.

Definition 10.1 A *language model* is a pair (V, L) where V is a finite set of basic elements of *vocabulary*, and $L : T \to V^+$ is a function from the class T of all types to the powerset of the set of all finite (non-empty) strings of elements from V inductively defined to satisfy the following postulates:

$$L(x \to y) = \{a : \forall b \in L(x),\ ba \in L(y)\} \qquad L(x \leftarrow y) = \{b : \forall a \in L(y),\ ba \in L(x)\}$$

$$L(x \circ y) = \{ab : a \in L(x) \text{ and } b \in L(y)\}$$

Where $L(x)$ represents the class of all language units of type x. Clearly this definition respects the interpretation.

We can then prove the following theorem which shows that the Gentzen calculus 'gets it right.'

Definition 10.2 Expand the definition of L to take lists of types as well as types by setting $L(x_1, x_2, \ldots, x_n) = L(x_1 \circ x_2 \circ \cdots \circ x_n)$. A sequent $X \rhd x$ is said to be *valid in the model* (V, L) if $L(X) \subseteq L(x)$.

THEOREM 10.1 *The sequent* $X \rhd x$ *is provable if and only if it is valid in each language model.*

Proof: The proof, first given by Buszkowski [21] uses standard techniques, and we only sketch it here.

For soundness, it suffices to note that each axiom is valid in each language model (trivial) and that the rules of the Gentzen calculus preserve validity in a language model (easy).

For completeness, we show that if $X \not\rhd x$ then there is a language model (V, L) in which it is not valid. This is reasonably simple. Take as elementary vocabulary a finite set T' of types such that $X \subseteq T'^{+}$, $x \in T'$, and T' is closed under subtypes. We take this set of types to be the basic vocabulary V of the language model (and we identify concatenation of basic vocabulary with the list formation in the antecedents of the sequents). We then take L to be defined by setting $L(x) = \{Y \in T'^{+} : Y \rhd x\}$ and show that this satisfies the inductive postulates on L. For example, consider $L(x \rightarrow y)$. $Y \in L(x \rightarrow y)$ iff $Y \in T'^{+}$ and $Y \rhd x \rightarrow y$. If $Z \in L(x)$ we have $Z \in T'^{+}$ and $Z \rhd x$. This gives $ZY \in T'^{+}$ and $ZY \rhd y$, giving $ZY \in L(y)$ as desired. Conversely, suppose that for all $Z \in L(x)$, $ZY \in L(y)$. This means that $xY \in L(y)$ (as $x \in L(x)$), giving $xY \rhd y$ and so $Y \rhd x \rightarrow y$ and $Y \in L(x \rightarrow y)$ as desired. $\lhd$

10.2 Conjunction and Disjunction

One problem in the analysis of sentence structure is the account given of parts of speech such as and, or and but. Assigning types to these words is difficult. In the sentence Jack and Jill walked and talked the two occurrences of and have types $NP \rightarrow NP \leftarrow NP$ and $(NP \rightarrow S) \rightarrow (NP \rightarrow S) \leftarrow (NP \rightarrow S)$ respectively. (It is a simple exercise in $\mathbf{L}$ to show that $(A \rightarrow B) \leftarrow C$ and $A \rightarrow (B \leftarrow C)$ are weakly equivalent to $A \circ C \rightarrow B$, so bracketing is unnecessary here.) In fact, while and always seems to have type $X \rightarrow X \leftarrow X$, there seems to be no limit to the number of different types that can be substituted for X. Oehrle gives some examples [104]

Kim gave and Hilary offered Whitney a documentary.	$(S \leftarrow NP) \leftarrow NP$
Kim gave Hilary and Sal offered Whitney a documentary.	$S \leftarrow NP$
Kim gave Whitney a documentary and Hilary a cartoon.	$NP \circ NP$
Kim gave Whitney and offered Hilary a documentary.	$NP \rightarrow S \leftarrow NP$

Operators such as and have caused difficulty for those seeking to give a unified account of syntactic types. One way to proceed is to take and to have the polymorphic type $X \rightarrow X \leftarrow X$, where X ranges over the class of all "conjoinable types." Another way to proceed is by taking and, or and but to be operators on a par with $\circ$, $\rightarrow$ and $\leftarrow$, and to create new rules for them. This seems to be a category mistake, as the type operators

$\rightarrow$, $\leftarrow$ and $\circ$ don't appear in our language. They are operations on types. Our study of **L** opens up another way, closely associated to both of these methods, but perhaps, better motivated. Given that a piece of syntax is of type A and of type B, it is sensible to take it to be of type $A \wedge B$. Similarly, if it is either of type A or of type B, then it is also of type $A \vee B$. Then, we can take and to be of type

$$\bigwedge_{X \in CT} X \rightarrow X \leftarrow X$$

where CT is the set of all conjoinable types. Of course, if CT is infinite (as it may be), we need to introduce infinitary conjunction (over only countable sets, thankfully) which is not difficult to define. So, we can extend language models as follows:

Definition 10.3 An *extended language model* is a language model (V, L) as before where L now maps the larger T of all types (including conjunctive and disjunctive types) to the powerset of the set V^+ of all finite (non-empty) strings of elements from V inductively defined to satisfy the following postulates:

$$L(x \rightarrow y) = \{a : \forall b \in L(x),\ ba \in L(y)\} \qquad L(x \leftarrow y) = \{b : \forall a \in L(y),\ ba \in L(x)\}$$

$$L(x \circ y) = \{ab : a \in L(x) \text{ and } b \in L(y)\}$$

$$L(x \wedge y) = L(x) \cap L(y) \qquad L(x \vee y) = L(x) \cup L(y)$$

Where $L(x)$ represents the class of all language units of type x. Again, this definition respects the interpretation.

The choice of the rule of disjunction has reflected a design consideration. We are not allowing the case where a vocabulary unit is assigned the type $x \vee y$ *without* it also being assigned either x or y. There may be certain applications in which such an assignment is useful. For example, in encoding partial type-assignments. We'll leave this consideration for another time, and only examine language models in which disjunction is *really* disjunction.

Independently of this work, Makoto Kanazawa [61] examined a way of extending the Lambek calculus to incorporate conjunction and disjunction. His proposal is to add the following rules to the Gentzen calculus

$$\frac{X \rhd x \quad X \rhd y}{X \rhd x \wedge y} \qquad \frac{X, x, Y \rhd z}{X, x \wedge y, Y \rhd z} \qquad \frac{X, x, Y \rhd z}{X, y \wedge x, Y \rhd z}$$

$$\frac{X, x, Y \rhd z \quad X, y, Y \rhd z}{X, x \vee y, Y \rhd z} \qquad \frac{X \rhd x}{X \rhd x \vee y} \qquad \frac{X \rhd x}{X \rhd y \vee x}$$

We'll call this system $\mathbf{L}^-_{\rightarrow\circ\leftarrow\wedge\vee}$ (and similarly $\mathbf{L}^-_{\rightarrow\circ\leftarrow\wedge}$, etc. for fragments). In his paper, Kanazawa proves some very interesting things about this extension of the Lambek calculus. Some of which we will reiterate later. However, it must be said that this extension is at best, a useful approximation to a true account of conjunctive and disjunctive types. This is because the distributive law

$$x \wedge (y \vee z) \ \rhd\ (x \wedge y) \vee (x \wedge z)$$

is not provable in the system. This is a shortcoming, because under the interpretation of the system (given by extended language models) the law comes out as valid. This is not a strange feature of the way in which extended models were defined, either. Rather, it is a feature of the way that conjunction and disjunction of types interact. If something is both of type x and either of type y or z, then it's either of type x and of y, or of type x and of type z.

So, a more faithful way to extend the Gentzen calculus to deal with conjunction and disjunction is by expanding the bunching operation to admit extensional (conjunctive) bunching as well as intensional (concatenating) bunching, in the manner of Chapter 2. In this way, we get a Gentzen system we've already studied, and it is one in which distribution is provable. As well, when we read X, Y as "something which is both of type X and of type Y" and $X; Y$ as "something of type X concatenated with something of type Y" then the axioms and rules of the usual Gentzen system (and requiring that intensional bunching be associative) are all true. Call the resulting system the *distributive extended Lambek calculus*, or $\mathbf{L}_{\rightarrow\circ\leftarrow\wedge\vee}$

Whether they are the *whole* truth about extended language models is an open question. I have not been able to prove that if $X \not\triangleright x$ (in the new Gentzen system) then there is an extended language model (V, L) in which $X \triangleright x$ is not valid. Disjunction adds a level of difficulty to the completeness proof. The class $L(x \vee y)$ may contain things that are not in either $L(x)$ or $L(y)$ as defined. Some way has to be found that can 'beef up' the classes while respecting the invalidity of the sequent in question. This looks rather difficult, and none of the priming lemmas from previous chapters seem to crack the problem. So, we leave it open, for further research. So, we have the following result:

THEOREM 10.2 *Kanazawa's extended calculus is sound, but not complete with respect to extended language models. It fails to validate the distributive law* $x \wedge (y \vee z) \triangleright (x \wedge y) \vee (x \wedge z)$*, which is valid in all language models. The distributive extended calculus is also sound with respect to extended language models.*

It is a reasonably simple task to show the following:

THEOREM 10.3 $\mathbf{L}^{-}_{\rightarrow\circ\leftarrow\wedge}$ *is the same as* $\mathbf{L}_{\rightarrow\circ\leftarrow\wedge}$*. Similarly for any fragments. That is, if we ignore disjunction, the distributive and non-distributive extended Lambek calculi are identical.*

Proof: It is sufficient to show that both calculi are the same given by the same Hilbert proof theory. This is given by adding the conjunction axioms

$$A \wedge B \rightarrow A \qquad A \wedge B \rightarrow B \qquad (A \rightarrow B) \wedge (A \rightarrow C) \rightarrow (A \rightarrow B \wedge C)$$

on top of the usual implication axioms. The result follows by methods used in earlier chapters. ◁

10.3 Recognising Power

The major result that Kanazawa proves about his extended calculus is that it improves recognizing power. Specifically, a language $\mathcal{L}$ is an arbitrary set of strings of symbols. A language model (V, L) is said to *recognize* the language $\mathcal{L}$ just when there is a type t such that $\mathcal{L} = L(t)$. As Kanazawa explains [61], the original Lambek calculus can recognize all context free languages. Whether this is *all* that can be recognized is a famous open problem in this area.

Kanazawa's major result is that his extended Lambek calculus has a strictly greater recognizing power. He proves the following result:

THEOREM 10.4 $\mathbf{L}^{-}_{\to\leftarrow\wedge}$ *recognises any finite intersection of* $\mathbf{L}_{\to\leftarrow}$ *recognizable languages.*

And as a corollary, since context-free languages are not closed under intersection, we have

THEOREM 10.5 $\mathbf{L}^{-}_{\to\leftarrow\wedge}$ *recognizes some non-context-free languages.*

We can shamelessly appropriate this result for our own systems, because $\mathbf{L}^{-}_{\to\leftarrow\wedge}$ and $\mathbf{L}_{\to\leftarrow\wedge}$ are identical. So, we finish this chapter with the following result.

COROLLARY $\mathbf{L}_{\to\leftarrow\wedge}$ *recognises any finite intersection of* $\mathbf{L}_{\to\leftarrow}$ *recognizable languages, and so, it recognizes some non-context-free languages.*

It is clear that $\mathbf{L}^{+}$ is more reasonable than $\mathbf{L}^{-}$ on interpretation, because the distribution law is true when applied to linguistic types. Kanazawa's approach, which is certainly formally interesting and proof theoretically rich, does not correctly model the logic of conjunction and disjunction when applied to types. So, linguistic types are another application for contraction-free logics with distribution. $\mathbf{L}^{+}$ has all of the good things of its distribution-lacking counterpart $\mathbf{L}^{-}$ (such as recognising power) together with one thing more. It is correct.

10.4 Note

[1] Refer to the bibliography of Oherle's paper [104] for references to the extensive literature on the Lambek calculus.

Chapter 11
Contraction-Free Arithmetic

One, Two, Three!
I love to count!
— THE COUNT *Sesame Street* [24]

11.1 Motivation

There's something suspicious about arithmetic in classical logic. Gödel's machinery, by coding formulae into the terms of the language, reveals something quite strange. Gödel has shown that it is possible to represent provability in the system in the language of the system. That is, if $\ulcorner A \urcorner$ is the Gödel numeral of the formula A, there is a predicate Prv expressible in the language such that

$$\vdash \mathrm{Prv}\ulcorner A \urcorner \text{ iff } \vdash A$$

$$\vdash \mathrm{Prv}\ulcorner A \to B \urcorner \to (\mathrm{Prv}\ulcorner A \urcorner \to \mathrm{Prv}\ulcorner B \urcorner)$$

$$\vdash \mathrm{Prv}\ulcorner A \urcorner \to \mathrm{Prv}\ulcorner \mathrm{Prv}\ulcorner A \urcorner \urcorner$$

So, Prv contains as its provable extension all of the provable sentences in arithmetic, and it interacts with implication in a standard way. It would be rational to expect that in a decent theory of provability, it would be possible to show that $\vdash \mathrm{Prv}\ulcorner A \urcorner \to A$. That is, we could show that if something is provable, it is true. This is impossible because of a result due to Löb [69]. In classical arithmetics,

$$\text{If } \vdash \mathrm{Prv}\ulcorner A \urcorner \to A \text{ then } \vdash A$$

This rather limits the possibility for the theoremhood of $\mathrm{Prv}\ulcorner A \urcorner \to A$. The proof is rather simple, and we sketch it here.

Suppose that $\vdash \mathrm{Prv}\ulcorner A \urcorner \to A$. The diagonal lemma (see below) ensures that there is a sentence C such that

$$\vdash C \leftrightarrow (\mathrm{Prv}\ulcorner C \urcorner \to A)$$

So by the properties of provability and the biconditional,

$$\vdash \mathrm{Prv}\ulcorner C \to (\mathrm{Prv}\ulcorner C \urcorner \to A) \urcorner$$

and so,

$$\vdash \mathrm{Prv}\ulcorner C \urcorner \to \mathrm{Prv}\ulcorner \mathrm{Prv}\ulcorner C \urcorner \to A \urcorner$$

by the distribution of Prv over implication. Distributing again, and by applying the fact that $\vdash \mathrm{Prv}\ulcorner \mathrm{Prv}\ulcorner C \urcorner \urcorner \to \mathrm{Prv}\ulcorner C \urcorner$ we have

$$\vdash \mathrm{Prv}\ulcorner C \urcorner \to (\mathrm{Prv}\ulcorner C \urcorner \to \mathrm{Prv}\ulcorner A \urcorner)$$

Classically speaking, we may infer that

$$\vdash \mathrm{Prv}\ulcorner C \urcorner \to \mathrm{Prv}\ulcorner A \urcorner$$

though the reader will be forgiven for suspecting that there is something fishy about this step. By our assumption that $\vdash \mathit{Prv}\ulcorner A \urcorner \to A$ we have

$$\vdash \mathit{Prv}\ulcorner C \urcorner \to A$$

which with our fact that $\vdash C \leftrightarrow (\mathit{Prv}\ulcorner C \urcorner \to A)$ yields

$$\vdash C$$

and hence, $\vdash \mathit{Prv}\ulcorner C \urcorner$, which by *modus ponens* gives

$$\vdash A$$

as required. From a fact about provability (that if A is provable, it is true) we are able to prove anything we like. A handy ability to have.

The problems do not stop with provability. The inferential moves of classical logic dictate that we cannot add a predicate 'T' into the language to represent truth in that language, if we wish the system to remain nontrivial. If we add as an axiom

$$\vdash T\ulcorner A \urcorner \leftrightarrow A$$

(where $\ulcorner A \urcorner$ is the Gödel numeral of the sentence A), we can prove anything we like. The diagonal lemma ensures that we have sentences L and C in the language that satisfy

$$\vdash \sim T\ulcorner L \urcorner \leftrightarrow L \qquad \vdash (T\ulcorner C \urcorner \to A) \leftrightarrow C$$

The standard deductions (which we saw way back in the introduction) will yield a contradiction in the first case, and A in the second case. Neither is a good thing to prove in the context of classical logic.

Perhaps one might defer to something mystic such as the 'undefinability' of truth, and so accept this result as a limitation on our ability to model truth in our formal systems. But this is profoundly unsatisfying, because there *is* such a thing as truth about arithmetic, and we can talk about it in the same language that we use to talk about arithmetic. Talk of truth and arithmetic ought to have a semantics given that we aren't talking nonsense.

So, classical arithmetic has a number of shortcomings. There is a lot of scope for the study of systems that don't have these shortcomings.

Clearly, contraction-free logics provide one way ahead. In each of the paradoxical deductions, a contraction related move was used. So, the way is open to consider arithmetics based on contraction-free logics. We will formulate a range of arithmetics as theories in each of our favourite logics. This will show us how going without contraction affects standard mathematical theories.

11.2 Definitions and Basic Results

Our language is first order, with function symbols

$$\underline{0} \quad ' \quad + \quad \times$$

where the first is nullary, the second unary, and the others binary.

In this language, we will add Peano's axioms to the logics **CK**, **R**, **C**, **EW**, **TW** and **DW** to give the arithmetics $\mathbf{CK}^\#$, $\mathbf{R}^\#$, $\mathbf{C}^\#$, $\mathbf{EW}^\#$, $\mathbf{TW}^\#$ and $\mathbf{DW}^\#$.

Definition 11.1 The arithmetics $\mathbf{X}^\#$ are given by adding axioms and rules to the predicate logic $\mathbf{X}$. These axioms and rules take two forms. Some (those that are free of conditionals) express arithmetic facts. The others (containing conditionals) express valid arithmetic deductions. In the context of contraction-free logics, these encode legitimate deductions, keeping track of the number of times premises are used. So, the axioms and one rule of arithmetic are

$$\begin{array}{rl}
\textit{Identity} & \underline{0} = \underline{0}\ [\text{Ident}],\quad \forall x \forall y(x = y \to y = x)\ [\text{Sym=}], \\
& \forall x \forall y \forall z\big(y = z \to (x = y \to x = z)\big)\ [\text{Trans=}], \\
& \forall x \forall y(x' = y' \to x = y)\ [{}'\text{E}], \\
\textit{Successor} & \forall x \forall y(x = y \to x' = y')\ [{}'\text{I}],\quad \forall x(\underline{0} \neq x')\ [\sim f^+], \\
\textit{Addition} & \forall x(x + \underline{0} = x)\ [+\underline{0}],\quad \forall x \forall y\big(x + y' = (x + y)'\big)\ [+'], \\
\textit{Multiplication} & \forall x(x\underline{0} = \underline{0})\ [\times\underline{0}],\quad \forall x \forall y(xy' = xy + x)\ [\times'], \\
\textit{Induction} & A(\underline{0}),\ \forall x\big(A(x) \to A(x')\big) \vdash \forall x A(x)\ [\text{PMI}].
\end{array}$$

where as usual, multiplication is abbreviated by juxtaposition. A question immediately arises: why this set of axioms? Clearly the addition and multiplication axioms are unproblematic as they stand. The induction axiom is weak, because it only holds in rule form. We will discuss this below. The other concern is with the implications in the successor axioms and the identity axioms. In these we have followed Bob Meyer's choice of axioms for his relevant arithmetic $\mathbf{R}^\#$. We take $\underline{0} = \underline{0}$ as an identity axiom because from it $[{}'\text{I}]$ and induction, we can prove $\forall x(x = x)$ anyway. The symmetry of identity follows from the intuition that $x = y$ and $y = x$ are exactly the same claim. For the transitivity of identity, note that our conditional records the fact that to deduce $x = z$ from $y = z$ and $x = y$ we had to use both of the identities once each. To claim it in the form $x = y \wedge y = z \to x = z$ would be to record something different, and in a context where $(A \to B) \wedge (B \to C) \to (A \to C)$ fails in general, there is no reason to think that the identity form will succeed. Note too that [Trans=] is one instance of the strong form of substitutivity of identities: $x = y \to \big(A(x) \to A(y)\big)$ which we argued in Chapter 4, need not hold in general in our logics. For this reason, we do not take strong substitutivity as an axiom — rather, we assume it in the atomic case, to see what form it takes in general.

And in general, from these axioms substitutivity is simple to prove in the rule form $x = y \vdash A(x) \leftrightarrow A(y)$ by induction on the base case where $A(x)$ is an identity. This is as one would expect, given the ruminations of Chapter 4.

The only things left to consider are the successor axioms. These indicate that from $x = y$ you can validly deduce $x' = y'$, and vice versa. In each case, using the antecedent only once. This seems quite straightforward. In this work we will consciously follow the policy of assuming as little as we need to get by. If, after future investigation we decide that stronger forms of the axioms are desirable, then well and good. None of this work is wasted, because it will follow under the stronger assumptions.

In natural deduction form the identity and induction rules can be expressed as

$$\frac{X \Vdash a = b}{X \Vdash b = a}\ [\text{Sym=}] \qquad \frac{X \Vdash a = b}{\overline{X \Vdash a' = b'}}\ [{}'\text{I},{}'\text{E}] \qquad \frac{X \Vdash a = b \quad Y \Vdash b = c}{X; Y \Vdash a = c}\ [\text{Trans=}]$$

$$\frac{0 \Vdash A(\underline{0}) \quad 0 \Vdash \forall x\big(A(x) \to A(x')\big)}{0 \Vdash \forall x A(x)} \quad \text{[PMI]}$$

These axioms tell us the arithmetic facts, and they licence some basic arithmetic inferences dealing with identity and the successor function. We will see how much can be derived from these simple facts and rules, even in the absence of contraction, and even in very weak logic like **DW**.

Firstly, some helpful notation.

Definition 11.2 For each number n, the term $\underline{n}$ is the *standard representation* of n. Defined recursively, $\underline{0} = 0$ (surprise!) and $\underline{n+1} = \underline{n}'$.

In an effort to situate our work with respect to the existing literature on relevant arithmetic, note that our definition of $\mathbf{R}^{\#}$ differs from Meyer's $\mathbf{R}^{\#}$ by using a weaker induction rule [89,90]. The reason for this departure from the norm is that the stronger induction principle used there

$$A(\underline{0}) \wedge \forall x\big(A(x) \to A(x')\big) \to \forall x A(x) \qquad \text{[pseudo PMI]}$$

looks too much like an extended case of *pseudo modus ponens* and conjunctive syllogism to be comfortable. Cashing out the quantifiers as extended conjunctions it the strong induction principle becomes.

$$A(\underline{0}) \wedge (A(\underline{0}) \to A(\underline{1})) \wedge (A(\underline{1}) \to A(\underline{2})) \wedge \cdots \;\to\; A(\underline{0}) \wedge A(\underline{1}) \wedge \cdots$$

At the very least, to get the consequent from the antecedent, we must use the antecedent much more than once. It is used once for $A(\underline{0})$, then another time to get $A(\underline{1})$ from $A(\underline{0})$, and another to get $A(\underline{2})$ from $A(\underline{1})$, and so on. The strong axiom does not encode this information at all, so it doesn't seem an appropriate way to formalise induction in a contraction-free context. We settle for the weaker principle PMI, which is to pseudo PMI as the real *modus ponens* is to *pseudo modus ponens*.

However, in the context of $\mathbf{R}$, and contraction-added logics in its vicinity, pseudo PMI is provable from PMI, so our $\mathbf{R}^{\#}$ coincides with Meyer's.

THEOREM 11.1 *If the logic* X *satisfies the structural rule* $(X;Y);X \Longleftarrow X;Y$ *(as* $\mathbf{R}$ *does) then the axiomatic form of induction is provable in* $X^{\#}$.

Proof: Given this structural rule, the deduction from $A \to (B \to C)$ to $(A \to B) \to (A \to C)$ (the rule form of *self distribution*) is admissible (as is easily verified by the reader.) Given this, and abbreviating $A(\underline{0}) \wedge \forall x\big(A(x) \to A(x')\big)$ as Φ we can proceed as follows:

1	(1)	Φ	A
1	(2)	$A(\underline{0})$	$1 \wedge$E
0	(3)	$\Phi \to A(\underline{0})$	1, 2 CP
1	(4)	$A(a) \to A(a')$	$1 \wedge$E, $\forall$E
0	(5)	$\Phi \to \big(A(a) \to A(a')\big)$	1, 4 CP
0	(6)	$\big(\Phi \to A(a)\big) \to \big(\Phi \to A(a')\big)$	5 Self Distribution
0	(7)	$\forall x\big(\big(\Phi \to A(x)\big) \to \big(\Phi \to A(x')\big)\big)$	$6 \forall$I
0	(8)	$\forall x\big(\Phi \to A(x)\big)$	3, 7 PMI
0	(9)	$\Phi \to \forall x A(x)$	8 Confinement

◁

Now we'll begin to see what is provable in these arithmetics. It will soon be seen that it's a surprising amount, and that even arithmetics as weak as $\mathbf{DW}^{\#}$ do not seem appreciably weaker than $\mathbf{C}^{\#}$, or even $\mathbf{R}^{\#}$. (Although, as we will also see, the case differs with $\mathbf{CK}^{\#}$.)

Firstly we will show that addition and multiplication are commutative, associative, and that they distribute in the usual way. The proofs are in a Lemmon-style natural deduction system. Each of our results will hold in each arithmetic, unless specified otherwise. The methods used in this chapter follow Slaney's work in contraction-free relevant arithmetic [149] (in which he shows that $\sqrt{2}$ is irrational, in even $\mathbf{TW}^{\#}$).

First, note that the order of composing identities when using Trans= is unimportant. In other words, if we can prove $A; B \vdash a = c$ from $A \vdash a = b$ and $B \vdash b = c$, we could just as well have proved $B \vdash c = b$ and $A \vdash b = a$, using Sym= and then deduced $B; A \vdash a = c$ using Trans= and Sym=. Now we may proceed with our proofs.

LEMMA 11.2 $\vdash \forall x \forall y(x + y' = x' + y)$

Proof:

0	(1)	$(a + \underline{0})' = a + \underline{0}'$	$+'$
0	(2)	$a + \underline{0} = a$	$+0$
0	(3)	$(a + \underline{0})' = a'$	$2\,'$I
0	(4)	$a' + \underline{0} = a'$	$+0$
0	(5)	$a + \underline{0}' = a' + \underline{0}$	1, 3, 4 Trans=, Sym=
0	(6)	$\forall x(x + \underline{0}' = x' + \underline{0})$	$5 \forall$I
7	(7)	$\forall x(x + b' = x' + b)$	A
7	(8)	$a + b' = a' + b$	$7 \forall$E
7	(9)	$(a + b')' = (a' + b)'$	$8\,'$I
7	(10)	$a + b'' = a' + b'$	$9 +'$, Trans=
7	(11)	$\forall x(x + b'' = x' + b')$	$10 \forall$I
0	(12)	$\forall x \forall y(x + y' = x' + y)$	6, 7, 11 PMI

◁

This is a typical proof of an $\to$-free formula in our system. To prove a statement like this, you rarely need to use the machinery for nested conditionality. The equational axioms and rules are enough. The only conditionals you use, like Trans= and Sym=, are first degree, and so, nesting them doesn't enter the picture — and so, the absence of contraction is irrelevant.

Another fact to notice is the way the proof uses induction. Typically a proof of a result like this first shows the result for one variable equal zero, then the induction step is proved, and PMI applied. Another example of this structure is the commutativity of addition.

LEMMA 11.3 $\vdash \forall x \forall y(x+y=y+x)$

Proof:

0	(1)	$\forall x(x+\underline{0}=x)$	$+0$
0	(2)	$\underline{0}+\underline{0}=\underline{0}$	$1\,\forall$E
3	(3)	$\underline{0}+a=a$	A
3	(4)	$(\underline{0}+a)'=a'$	$3\,'$I
0	(5)	$(\underline{0}+a)'=\underline{0}+a'$	$+'$
3	(6)	$\underline{0}+a'=a'$	4, 5 Trans=
0	(7)	$\forall x(\underline{0}+x=x)$	2, 3, 6 PMI
0	(8)	$\forall x(\underline{0}+x=x+\underline{0})$	1, 7 Fiddle
9	(9)	$\forall x(a+x=x+a)$	A
9	(10)	$a+b=b+a$	$9\,\forall$E
0	(11)	$a'+b=a+b'$	Lemma 11.2
0	(12)	$a+b'=(a+b)'$	$+'$
9	(13)	$(a+b)'=(b+a)'$	$10\,'$I
0	(14)	$(b+a)'=b+a'$	$+'$
9	(15)	$a'+b=b+a'$	11–15 Trans=
9	(16)	$\forall x(a'+x=x+a')$	$15\,\forall$I
0	(17)	$\forall x \forall y(x+y=y+x)$	8, 9, 16 PMI

$\triangleleft$

We'll use this fact in other natural deduction proofs. When we do, we'll annotate it 'Com+.' $\triangleleft$

Associativity of addition is another famous equational result. It is retained in each of our arithmetics.

LEMMA 11.4 $\vdash \forall x \forall y \forall z\big(x+(y+z)=(x+y)+z\big)$

Proof:

0	(1)	$b+\underline{0}=b$	$+0$
0	(2)	$a+(b+\underline{0})=a+b$	$1\,+$I
0	(3)	$a+b=(a+b)+\underline{0}$	$+0$
0	(4)	$a+(b+\underline{0})=(a+b)+\underline{0}$	2, 3 Trans=
0	(5)	$\forall x \forall y\big(x+(y+\underline{0})=(x+y)+\underline{0}\big)$	$4\,\forall$I
6	(6)	$\forall x \forall y\big(x+(y+c)=(x+y)+c\big)$	A
6	(7)	$a+(b+c)=(a+b)+c$	$6\,\forall$E
0	(8)	$(a+b)+c'=\big((a+b)+c\big)'$	$+'$
6	(9)	$\big((a+b)+c\big)'=\big(a+(b+c)\big)'$	7 Trans=, $'$I
0	(10)	$\big(a+(b+c)\big)'=a+(b+c)'$	$+'$
0	(11)	$a+(b+c)'=a+(b+c')$	$+'$, $+$I
6	(12)	$(a+b)+c'=a+(b+c')$	8, 9, 10, 11 Trans=
6	(13)	$\forall x \forall y\big(x+(y+c')=(x+y)+c'\big)$	12 Trans=, $\forall$I
0	(14)	$\forall x \forall y \forall z\big(x+(y+z)=(x+y)+z\big)$	5, 6, 13 PMI

$\triangleleft$

Not all interesting results are quantified equations like these. Given that we have a conditional that records facts about deductions, we may be interested in seeing exactly what kinds of inferences are validated in our arithmetics. One example is a generalisation of the rule $'$I.

LEMMA 11.5 $\vdash \forall x \forall y \forall z(y = z \to y + x = z + x)$

Proof:

1	(1)	$a = b$	A
0	(2)	$a + \underline{0} = a$	$+0$
0	(3)	$b + \underline{0} = b$	$+0$
1	(4)	$a + \underline{0} = b + \underline{0}$	1, 2, 3 Trans=
0	(5)	$a = b \to a + \underline{0} = b + \underline{0}$	1, 4 CP
6	(6)	$a = b \to a + c = b + c$	A
1; 6	(7)	$a + c = b + c$	1, 6 MP
1; 6	(8)	$(a + c)' = (b + c)'$	7 $'$I
1; 6	(9)	$a + c' = b + c'$	8 $+'$, Trans=
6	(10)	$a = b \to a + c' = b + c'$	1, 9 CP
0	(11)	$\forall x(a = b \to a + x = b + x)$	5, 6, 10 PMI
0	(12)	$\forall x \forall y \forall z(y = z \to y + x = z + x)$	11 $\forall$I

◁

The form of this proof is instructive. Firstly, the base case of the induction is simple. It is simple to prove that

$$\vdash a = b \to a + \underline{0} = b + \underline{0}$$

rather trivially. Next is the induction step. We need to show that

$$\vdash \big(a = b \to a + c = b + c\big) \to \big(a = b \to a + c' = b + c'\big)$$

which is given on line 10. But given the fact that $(a + c)' = a + c'$ and $(b + c)' = b + c'$ this is just rule prefixing on the instance

$$\vdash a + c = b + c \to (a + c)' = (b + c)'$$

of $'$I. The nesting of conditionals here works in very weak logics because at the crucial step (applying prefixing) only the rule version is necessary — because the result prefixed, $'$I, is a theorem, and not merely an assumption. Otherwise, we may not have had the resources (in $\mathbf{DW}^{\#}$ at least) to apply prefixing in axiom form.

Or in other words, since $'$I is a *logical* deduction according to our axioms, the result of iterating it is still a logical deduction, by rule transitivity. We have

$$\vdash a = b \to a' = b'$$

$$\vdash a' = b' \to a'' = b''$$

$$\vdash a'' = b'' \to a''' = b'''$$

$$\vdots \qquad\qquad \vdots$$

Composing the right number of these conditionals will give us the desired instance of the universal claim. (Given some easily provable facts about the relationship between successor and addition.)

Commutativity of addition gives us the obvious corollary to this result.

COROLLARY $\vdash a = b \to c + a = c + b$

Proof:

1	(1)	$a = b$	A
1	(2)	$a + c = b + c$	1 +I
0	(3)	$a + c = c + a$	Com+
0	(4)	$b + c = c + b$	Com+
1	(5)	$c + a = c + b$	2, 3, 4 Trans=
0	(6)	$a = b \to c + a = c + b$	1, 5 CP

◁

This result and the one before are called '+I' when used in natural deduction proofs, because they introduce an addition. The next result begins to show the differences between contraction-free arithmetics and their contraction added cousins.

COROLLARY $\vdash (a = b) \circ (c = d) \to a + c = b + d$

Proof:

1	(1)	$a = b$	A
1	(2)	$a + c = b + c$	1 +I
3	(3)	$c = d$	A
3	(4)	$b + c = b + d$	3 +I
1; 3	(5)	$a + c = b + d$	2, 4 Trans=
0	(6)	$(a = b) \circ (c = d) \to a + c = b + d$	1, 3, 5 CP

◁

This result is called $+I^2$. Here we have the first inklings of how contraction-freedom enters the picture. We have used both $a = b$ and $c = d$ to infer $a + c = b + d$, and our conditional has to record that fact. We couldn't prove (in this way) $(a = b) \wedge (c = d) \to a + c = b + d$, because this would not respect the fact that both $a = b$ and $c = d$ are used in the deduction.

Just as +I generalises ′I, it is simple to generalise ′E to get +E.

LEMMA 11.6 $\vdash \forall x \forall y \forall z (y + x = z + x \to y = z)$

Proof:

1	(1)	$a + \underline{0} = b + \underline{0}$	A
0	(2)	$a + \underline{0} = a$	+0
0	(3)	$b + \underline{0} = b$	+0
1	(4)	$a = b$	1, 2, 3 Trans=
0	(5)	$a + \underline{0} = b + \underline{0} \to a = b$	1, 4 CP
6	(6)	$a + c = b + c \to a = b$	A
7	(7)	$a + c' = b + c'$	A
0	(8)	$a + c' = (a + c)'$	+′
0	(9)	$b + c' = (b + c)'$	+′
7	(10)	$(a + c)' = (b + c)'$	7, 8, 9 Trans=
7	(11)	$a + c = b + c$	′E
7; 6	(12)	$a = b$	11, 6 MP
6	(13)	$a + c' = b + c' \to a = b$	7, 12 CP
0	(14)	$\forall x (a + x = b + x \to a = b)$	5, 6, 13 PMI
0	(15)	$\forall x \forall y \forall z (y + x = z + x \to y = z)$	14 ∀I

◁

This result has the obvious commuted version, which we leave to the reader to prove.

Let's go on to consider multiplication. Commutativity is but a simple 16 steps away. Note that these proofs don't use conditionality to any great extent — so they are straightforward in *any* of our systems.

Lemma 11.7 $\vdash \forall x \forall y(xy = yx)$

Proof:

0	(1)	$\forall x(x\underline{0} = \underline{0})$	$\times 0$
0	(2)	$\underline{0}\,\underline{0} = \underline{0}$	1 $\forall$E
3	(3)	$\underline{0}a = \underline{0}$	A
0	(4)	$\underline{0}a' = \underline{0}a + \underline{0}$	$\times'$
3	(5)	$\underline{0}a + \underline{0} = \underline{0} + \underline{0}$	3 +I
3	(6)	$\underline{0}a' = \underline{0}$	4, 5 Trans=, +0
0	(7)	$\forall x(\underline{0}x = \underline{0})$	2, 3, 6 PMI
0	(8)	$\forall x(\underline{0}x = x\underline{0})$	1, 7 Fiddle
9	(9)	$\forall x(ax = xa)$	A
9	(10)	$ab = ba$	9 $\forall$E
0	(11)	$a'b = ab + b$	$+'$
9	(12)	$ab + b = ba + b$	10 +I
0	(13)	$ba + b = ba'$	$\times'$
9	(14)	$a'b = ba'$	11, 12, 13 Trans=
9	(15)	$\forall x(a'x = xa')$	14 $\forall$I
0	(16)	$\forall x \forall y(xy = yx)$	8, 9, 15 PMI

◁

Then proving the distribution of multiplication over addition is relatively simple.

Lemma 11.8 $\vdash \forall x \forall y \forall z\big((x + y)z = xz + yz\big)$

Proof:

0	(1)	$\forall x \forall y\big((x + y)\underline{0} = x\underline{0} + y\underline{0}\big)$	Fiddle
2	(2)	$\forall x \forall y\big((x + y)c = xc + yc\big)$	A
2	(3)	$(a + b)c = ac + bc$	2 $\forall$E
0	(4)	$(a + b)c' = (a + b)c + (a + b)$	$+'$
0	(5)	$ac' = ac + a$	$\times'$
0	(6)	$bc' = bc + b$	$\times'$
0	(7)	$ac' + bc' = (ac + a) + (bc + b)$	5, 6 $+\mathrm{I}^2$
0	(8)	$(ac + a) + (bc + b) = (ac + bc) + (a + b)$	Additive Fiddle
2	(9)	$(a + b)c + (a + b) = (ac + bc) + (a + b)$	3 +I
2	(10)	$(a + b)c' = ac' + bc'$	4, 7, 8, 9 Trans=
2	(11)	$\forall x \forall y\big((x + y)c' = xc' + yc'\big)$	10 $\forall$I
0	(12)	$\forall x \forall y \forall z\big((x + y)z = xz + yz\big)$	1, 2, 11 PMI

◁

We call this 'Right Distribution.'

Corollary $\vdash \forall x \forall y \forall z\big(x(y + z) = xy + xz\big)$

Proof:

0	(1)	$a(b+c) = (b+c)a$	Com+
0	(2)	$ab = ba$	Com$\times$
0	(3)	$ac = ca$	Com$\times$
0	(4)	$ab + ac = ba + ca$	$2, 3 +\mathrm{I}^2$
0	(5)	$(b+c)a = ba + ca$	Right Distribution
0	(6)	$a(b+c) = ab + ac$	1, 4, 5 Trans=
0	(7)	$\forall x \forall y \forall z\big(x(y+z) = xy + xz\big)$	6 $\forall$I

◁

And this is 'Left Distribution.'

This brings with it the associativity of multiplication. Getting the proof is a little tricky, in that we have to use the distribution of addition over multiplication to get the induction step going. We present the proof for completeness sake.

LEMMA 11.9 $\vdash \forall x \forall y \forall z\big(x(yz) = (xy)z\big)$

Proof:

0	(1)	$b\underline{0} = \underline{0}$	$\times 0$
0	(2)	$a(b\underline{0}) = a\underline{0}$	1 $\times$I
0	(3)	$a\underline{0} = \underline{0}$	$\times 0$
0	(4)	$(ab)\underline{0} = \underline{0}$	$\times 0$
0	(5)	$a(b\underline{0}) = (ab)\underline{0}$	2, 3, 4 Trans=
0	(6)	$\forall x \forall y\big(x(y\underline{0}) = (xy)\underline{0}\big)$	5 $\forall$I
7	(7)	$\forall x \forall y\big(x(yc) = (xy)c\big)$	A
7	(8)	$a(bc) = (ab)c$	7 $\forall$E
0	(9)	$bc' = bc + b$	$\times'$
0	(10)	$a(bc') = a(bc + b)$	9 $\times$I
0	(11)	$a(bc + b) = a(bc) + ab$	10 Left Distribution
7	(12)	$a(bc) + ab = (ab)c + ab$	8 +I
0	(13)	$(ab)c' = (ab)c + ab$	$\times'$
7	(14)	$a(bc') = (ab)c'$	10, 11, 12, 13 Trans=
7	(15)	$\forall x \forall y\big(x(yc') = (xy)c'\big)$	14 $\forall$I
0	(16)	$\forall x \forall y \forall z\big(x(yz) = (xy)z\big)$	6, 7, 15 PMI

◁

Now consider the possibilities for a multiplication introduction rule akin to the addition introduction rule +I. At the very least we can prove the rule form given below.

LEMMA 11.10 $a = b \vdash \forall x(ax = bx)$.

Proof:

0	(1)	$a = b$	Hyp
0	(2)	$\underline{0} = \underline{0}$	Identity
0	(3)	$a\underline{0} = \underline{0}$	$\times 0$
0	(4)	$b\underline{0} = \underline{0}$	$\times 0$
0	(5)	$a\underline{0} = b\underline{0}$	2, 3, 4 Trans=
6	(6)	$ac = bc$	A
0	(7)	$ac' = ac + a$	$\times'$
6	(8)	$ac + a = bc + b$	1, 6 +I
0	(9)	$bc' = bc + b$	$\times'$
6	(10)	$ac' = bc'$	7, 8, 9 Trans=
0	(11)	$\forall x(ax = bx)$	5, 6, 10 PMI

◁

This result gives us $a = b \vdash f(a) = f(b)$ for arbitrary functions f. The result is a simple induction on the construction of f.

LEMMA 11.11 $a = b \vdash f(a) = f(b)$ *for any function* f.

Proof: By induction on the construction of f. Clearly the atomic cases all hold. For the induction step, note that if $a = b \vdash f_1(a) = f_1(b)$ and $a = b \vdash f_2(a) = f_2(b)$, then $f_1(a) = f_1(b) \vdash f_1(a)' = f_2(b)'$, $f_1(a) = f_1(b), f_2(a) = f_2(b) \vdash f_1(a) + f_2(a) = f_1(b) + f_2(b)$ and $f_1(a) = f_1(b), f_2(a) = f_2(b) \vdash f_1(a)f_2(a) = f_1(b)f_2(b)$, so the transitivity of weak validity gives us the result. ◁

Is there any better than weak validity for recording the facts about multiplication? The difficulty is in recording the number of times $a = b$ is used. If $a = b$ is hypothesised, as in the proof given above, you can use it as many times as you please without recording it — provided that you're happy with the resulting rule being only weakly valid. To get a strongly valid conditional we need to examine how many times $a = b$ is used in a proof of $ac = bc$. But this is simple. It is used c times. Firstly, we can prove this:

LEMMA 11.12 $\vdash (a = b) \circ (ac = bc) \to ac' = bc'$

Proof:

1	(1)	$a = b$	A
2	(2)	$ac = bc$	A
0	(3)	$ac' = ac + a$	$\times'$
0	(4)	$bc' = bc + b$	$\times'$
1; 2	(5)	$ac + a = bc + b$	1, 2 +I^2
1; 2	(6)	$ac' = bc'$	3, 4, 5 Trans=
0	(7)	$a = b \circ ac = bc \to ac' = bc'$	1, 2, 6 CP

◁

Then we get $\vdash (a = b) \circ (a = b) \to a\underline{2} = a\underline{2}$. So $\vdash (a = b) \circ (a\underline{2} = a\underline{2}) \to a\underline{3} = a\underline{3}$ gives $\vdash (a = b) \circ \big((a = b) \circ (a = b)\big) \to a\underline{3} = a\underline{3}$, or more simply, $\vdash (a = b)^3 \to a\underline{3} = a\underline{3}$. (Recall the notation introduced in Definition 2.5 in Chapter 2, for multiple fusions.) More generally, it is easy to show that

THEOREM 11.13 *For each* n, $\vdash (a = b)^n \to a\underline{n} = b\underline{n}$.

Proof: Apply Lemma 11.12 repeatedly. ◁

Now we move to consider how many arithmetic facts are proved by each of our arithmetics.

LEMMA 11.14 *All arithmetics prove* $\underline{n} + \underline{m} = \underline{n+m}$ *and* $\underline{n} \times \underline{m} = \underline{n \times m}$.

Proof: Induction on m. In $\mathbf{DW}^{\#}$ we can prove: $\underline{n} + \underline{0} = \underline{n+0} = \underline{n}$. Also, if we have $\underline{n} + \underline{m} = \underline{n+m}$, then we can prove: $\underline{n} + \underline{m+1} = \underline{n} + \underline{m}' = (\underline{n} + \underline{m})' = (\underline{n+m})' = \underline{n+m+1}$. The case for multiplication is similar. ◁

THEOREM 11.15 *All of our arithmetics prove all of the true numerical equations and inequations.*

Proof: Given an equation $t_1 = t_2$, we have the moves to reduce the terms t_1 and t_2 to their standard representations. If the equation is true under interpretation, the standard representations are identical, and the equation is provable. Otherwise, the standard representations of t_1 and t_2 will differ, and so, successive applications of $'$E will prove an equation of the form $\underline{0} = a'$ or $a' = \underline{0}$, which is false. *Modus tollens* gives the falsity of the original equation. So, a true inequation will be provable. ◁

Definition 11.3 f^+ is the sentence $\exists x(x' = \underline{0})$. We will often be able to use $A \to f^+$ as a 'positive' surrogate for $\sim A$, because of the following result:

LEMMA 11.16 $\vdash (A \to f^+) \to (t \to \sim A)$

Proof: $\vdash \sim f^+$, so $\vdash t \to \sim f^+$, and $\vdash (A \to f^+) \to (\sim f^+ \to \sim A)$ with $\vdash (\sim f^+ \to \sim A) \to (t \to \sim A)$ gives $\vdash (A \to f^+) \to (t \to \sim A)$. ◁

Now in $\mathbf{EW}^{\#}$, $\mathbf{C}^{\#}$, $\mathbf{R}^{\#}$ and $\mathbf{CK}^{\#}$ this gives $\vdash (A \to f^+) \to \sim A$ (as in these systems $\vdash (t \to \sim A) \to \sim A$, but in weaker systems this doesn't always hold).

Definition 11.4 $a < b$ is shorthand for $\exists x(b = a + x')$.

LEMMA 11.17 $\vdash a < b \to (a = b \to f^+)$, *and* $\vdash a = b \to (a < b \to f^+)$.
Proof:

1	(1)	$a < b$	A
2	(2)	$a = b$	A
1	(3)	$b = a + c'$	1 ∃E
2	(4)	$b + c' = a + c'$	2 +I
2; 1	(5)	$b = b + c'$	3, 4 Trans=
2; 1	(6)	$b + \underline{0} = b + c'$	$b = b + \underline{0}$ 5, Trans=
2; 1	(7)	$\underline{0} = c'$	6 +E
2; 1	(8)	f^+	7 ∃I
1	(9)	$a = b \to f^+$	2, 8 CP
0	(10)	$a < b \to (a = b \to f^+)$	1, 9 CP

◁

To show that $\vdash a = b \to (a < b \to f^+)$ use exactly the same proof, but on line 5, perform the Trans= move with the assumptions 1 and 2 permuted to give $a < b; a = b \Vdash b = b + c'$. Then conditional proof applied on line 8 will discharge $a < b$ instead of $a = b$, giving the appropriate conclusion. ◁

We can prove a number of strong facts about ordering. For example, here we prove that every number is either zero or a successor.

LEMMA 11.18 $\vdash \forall y(\underline{0} = y \lor \exists x(y = x'))$

Proof:

0	(1)	$\underline{0} = \underline{0} \lor \exists x(a = x')$	Identity, $\lor$I
0	(2)	$\exists x(a' = x')$	$a' = a'$, $\exists$I
0	(3)	$\underline{0} = a' \lor \exists x(a' = x')$	2 $\exists$I
0	(4)	$\underline{0} = a \lor \exists x(a = x') \supset \underline{0} = a' \lor \exists x(a' = x')$	3 $\supset$I
0	(5)	$\forall y(\underline{0} = y \lor \exists x(y = x'))$	1, 4 PMI $\triangleleft$

This can then be used to show that the only way for a to be less than b' is for a to be either b or strictly less than b — as one would expect.

LEMMA 11.19 $\vdash a < b' \leftrightarrow (a = b \lor a < b)$

Proof:

1	(1)	$a < b'$	A
1	(2)	$a + c' = b'$	1 $\exists$E
1	(3)	$(a + c)' = b'$	2 $+'$
1	(4)	$a + c = b$	3 $'$E
0	(5)	$\underline{0} = c \lor \exists x(c = x')$	Theorem
6	(6)	$\underline{0} = c$	A
6	(7)	$a + c = a + \underline{0}$	6 +I
1; 6	(8)	$a = b$	4, 7 Trans=, +0
1; 6	(9)	$a = b \lor a < b$	8 $\lor$I
10	(10)	$\exists x(c = x')$	A
10	(11)	$c = d'$	10 $\exists$E
10	(12)	$a + c = a + d'$	11 +I
1; 10	(13)	$b = a + d'$	4, 12 Trans=
1; 10	(14)	$a < b$	13 $\exists$I
1; 10	(15)	$a = b \lor a < b$	14 $\lor$I
1; 0	(16)	$a = b \lor a < b$	5, 6, 9, 10, 15 $\lor$E
0	(17)	$a < b' \to (a = b \lor a < b)$	1, 16 CP

1	(1)	$a = b \lor a < b$	A
2	(2)	$a = b$	A
0	(3)	$b' = b + \underline{0}'$	Fact
2	(4)	$b' = a + \underline{0}'$	2, 3 Fiddle
2	(5)	$a < b'$	4 $\exists$I
6	(6)	$a < b$	A
6	(7)	$b = a + c'$	6 $\exists$E
6	(8)	$b' = a + c''$	7 Fiddle
6	(9)	$a < b'$	8 $\exists$I
1	(10)	$a < b'$	1, 2, 5, 6, 9 $\lor$E
0	(11)	$(a = b \lor a < b) \to a < b'$	1, 10 CP $\triangleleft$

LEMMA 11.20 $\vdash a < b \leftrightarrow a' < b'$.

Proof:

1	(1)	$a < b$	A
1	(2)	$b = a + c'$	1 ∃E
1	(3)	$b' = (a + c')'$	'I
1	(4)	$b' = a' + c'$	3 Com+, +'
1	(5)	$a' < b'$	4 ∃I

1	(1)	$a' < b'$	A
1	(2)	$b' = a' + c'$	1 ∃E
1	(3)	$b' = (a + c')'$	2 Com+, +'
1	(4)	$b = a + c'$	'E
1	(5)	$a < b$	4 ∃I

◁

Sometimes it is useful to implement a kind of double induction. For this we use the enthymematic $\supset$, defined by taking $A \supset B$ to be $A \wedge t \to B$. This means that $A \vdash B \supset A$.

LEMMA 11.21 $A(\underline{0}, \underline{0}), \forall x A(\underline{0}, x'), \forall x A(x', \underline{0}), \forall x \forall y \big(A(x, y) \supset A(x', y')\big) \vdash \forall x \forall y A(x, y)$.

Proof:

0	(1)	$A(\underline{0}, \underline{0})$	Hyp
0	(2)	$A(\underline{0}, a')$	Hyp
0	(3)	$A(\underline{0}, a) \supset A(\underline{0}, a')$	2 ⊃I
0	(4)	$\forall y A(\underline{0}, y)$	1, 3 PMI
0	(5)	$A(a', \underline{0})$	Hyp
0	(6)	$\forall y A(a, y) \supset A(a', \underline{0})$	5 ⊃I
0	(7)	$\forall y A(a, y) \supset A(a', b')$	Hyp
0	(8)	$\big(\forall y A(a, y) \supset A(a', b)\big) \supset \big(\forall y A(a, y) \supset A(a', b')\big)$	7 ⊃I
0	(9)	$\forall y \big(\forall y A(a, y) \supset A(a', y)\big)$	6, 8 PMI
0	(10)	$\forall y A(a, y) \supset \forall y A(a', y)$	9 Confinem't
0	(11)	$\forall x \forall y A(x, y)$	4, 10 PMI

◁

Using double induction we can prove that $<$ is a total order on integers in this strong sense:

THEOREM 11.22 *Trichotomy:* $\vdash (a < b) \vee (a = b) \vee (a > b)$

This theorem sets our arithmetics apart from intuitionistic or constructive arithmetics — any two numbers are comparable, as you would expect.

Proof: The proof proceeds by a double induction.

0	(1)	$\underline{0} < a'$	Fact
0	(2)	$(\underline{0} < a) \vee (\underline{0} = a)$	1 Lemma 11.19
0	(3)	$(\underline{0} < a) \vee (\underline{0} = a) \vee (\underline{0} > a)$	2 $\vee$I
4	(4)	$(a < b) \vee (a = b) \vee (a > b)$	A
5	(5)	$a < b$	A
5	(6)	$a' < b'$	5 Lemma 11.20
5	(7)	$(a' < b') \vee (a' = b') \vee (a' > b')$	6 $\vee$I
8	(8)	$a = b$	A
8	(9)	$a' = b'$	8 $'$I
8	(10)	$(a' < b') \vee (a' = b') \vee (a' > b')$	9 $\vee$I
11	(11)	$a > b$	A
11	(12)	$a' > b'$	11 Lemma 11.20
11	(13)	$(a' < b') \vee (a' = b') \vee (a' > b')$	12 $\vee$I
4	(14)	$(a' < b') \vee (a' = b') \vee (a' > b')$	4, 5, 7, 8, 10, 11, 13 $\vee$E
0	(14)	$(a < b) \vee (a = b) \vee (a > b)$	4, 14 Double Induction ◁

We'll now head for another result that sets contraction-free arithmetics apart from intuitionistic arithmetic. We will prove excluded middle for all $\to$-free formulae. First a quick lemma.

Lemma 11.23 $\vdash \forall y \forall z \exists x (y = z \leftrightarrow \underline{0} = x)$

Proof:

0	(1)	$\underline{0} = a \leftrightarrow \underline{0} = a$	Theorem of logic
0	(2)	$a = \underline{0} \leftrightarrow \underline{0} = a$	Sym=
0	(3)	$\exists x(\underline{0} = a \leftrightarrow \underline{0} = x)$	1 $\exists$I
0	(4)	$\exists x(a = \underline{0} \leftrightarrow \underline{0} = x)$	2 $\exists$I
0	(5)	$\forall y \exists x(\underline{0} = y \leftrightarrow \underline{0} = x)$	3 $\forall$I
0	(6)	$\forall y \exists x(y = \underline{0} \leftrightarrow \underline{0} = x)$	4 $\forall$I
7	(7)	$\exists x(a = b \leftrightarrow \underline{0} = x)$	A
0	(8)	$a = b \leftrightarrow a' = b'$	$'$I, $'$E
7	(9)	$\exists x(a' = b' \leftrightarrow \underline{0} = x)$	7, 8 SE
0	(10)	$\forall y \forall z \exists x(y = z \leftrightarrow \underline{0} = x)$	5, 6, 7, 9 Double Induction ◁

Lemma 11.24 $\vdash \forall x(\underline{0} = x \vee \underline{0} \neq x)$

Proof:

0	(1)	$\forall x(\underline{0} = x \vee \exists y(x = y'))$	Lemma 11.18
0	(2)	$\underline{0} = b \vee \exists y(b = y')$	1 $\exists$E
3	(3)	$\underline{0} = b$	A
3	(4)	$\underline{0} = b \vee \underline{0} \neq b$	3 $\vee$I
5	(5)	$\exists y(b = y')$	A
5	(6)	$b = c'$	5 $\exists$E
0	(7)	$\underline{0} \neq c'$	$\sim f^+$
5	(8)	$\underline{0} \neq b$	6, 7 Subst
5	(9)	$\underline{0} = b \vee \underline{0} \neq b$	8, $\vee$I
0	(10)	$\underline{0} = b \vee \underline{0} \neq b$	2, 3, 4, 5, 9 $\vee$E ◁

These two lemmas help us prove the next strong result.

THEOREM 11.25 $\vdash A \vee \sim A$ *where the only connectives in* A *are* $\wedge$, $\vee$, $\sim$, $\forall$ *and* $\exists$.

Proof: By induction on the complexity of A.

- If A is an equation, then Lemma 11.23 shows us that it is equivalent to an equation of the form $\underline{0} = a$ and Lemma 11.24 shows that these satisfy excluded middle.
- If A is $\sim B$ and $\vdash B \vee \sim B$ then $\vdash \sim B \vee \sim\sim B$ as $B \vdash \sim\sim B$.
- If A is $B \wedge C$ where $\vdash B \vee \sim B$ and $\vdash C \vee \sim C$ we must have $\vdash (B \vee \sim B) \wedge (C \vee \sim C)$ so $\vdash (B \wedge C) \vee \big((B \wedge \sim C) \vee (\sim B \wedge C) \vee (\sim B \wedge \sim C)\big)$, which gives $\vdash (B \wedge C) \vee \sim(B \wedge C)$.
- Negation and conjunction steps give disjunction as $\vdash B \vee C \leftrightarrow \sim(\sim B \wedge \sim C)$.
- If A is $\forall x B(x)$ and $\vdash B(a) \vee \sim B(a)$ for each a, then $\vdash \forall x\big(B(x) \vee \sim B(x)\big)$ and so as $\sim B(a) \vdash \sim\forall x B(x)$ we have $\vdash \forall x\big(B(x) \vee \sim\forall x B(x)\big)$. Confinement gives $\vdash \forall x B(x) \vee \sim\forall x B(x)$.
- Negation and the universal quantifier steps give the existential case as $\vdash \exists x B(x) \leftrightarrow \sim\forall x \sim B(x)$. ◁

This result *underlines* the difference between these arithmetics and other non-classical arithmetics. It also helps us sharpen our attitude to excluded middle. In its place, there is nothing wrong with an instance of excluded middle. The Peano axioms give us enough to prove many instances of it. However, it doesn't follow as a matter of *logic* but as a matter of arithmetic. If we extend the arithmetic with other things such as a T-predicate, excluded middle may fail for propositions involving these extra connectives.

We have shown that the absence of contraction in itself is not crippling to an arithmetic theory. We get as much contraction as we need from the arithmetic axioms themselves. The result is even more striking in **CK**.

COROLLARY **CK**$^{\#}$ *is classical Peano arithmetic.*

Proof: Adding excluded middle to **CK** gives classical logic, so proving excluded middle in **CK**$^{\#}$ is enough to give us the result. To do this, we need to push through the $\rightarrow$ step in the induction in the proof above. This immediately follows the **CK** theorem $(A \vee \sim A) \circ (B \vee \sim B) \rightarrow (A \rightarrow B) \vee \sim(A \rightarrow B)$. ◁

This means that leaving contraction out has no effect on the arithmetic. This is a good result, because it shows that anything you can do in Peano arithmetic you can do in ways that are acceptable to **CK** scruples. The contraction that is present is there only because of the arithmetic axioms. If we add an extra predicate to the language (such as a truth predicate) this need not obey the strong classical strictures — in only has to obey the liberal laws of **CK**. So, this result has not shown us that every arithmetic theory in **CK** is fully classical. All we have shown is that the purely arithmetic parts of that theory are classical. If we abandon contraction to allow the presence of self-reference and truth, we need not worry that our arithmetic is weakened.

Now we'll present another result that shows that the extensional fragment of our weaker number theories also closely model classical Peano arithmetic. The main obstacle to showing that the extensional fragment of these theories is *identical* to classical arithmetic is the inadmissibility of γ in some of these theories. Meyer and Friedman have shown [45] that in $\mathbf{R}^{\#}$, there are extensional sentences A and $\sim A \vee B$ that are provable in $\mathbf{R}^{\#}$ such that B is not provable. Because of this, $\mathbf{R}^{\#}$ does not contain classical Peano arithmetic, because classical Peano arithmetic *is* closed under this rule (called γ in the literature). One offending formula that is a theorem of classical Peano arithmetic but not $\mathbf{R}^{\#}$ is

$$\forall x \exists y \forall z \exists u \exists v \big((\underline{2}x + \underline{1})u + yv = \underline{1} + z^2 v \big)$$

So, each of our arithmetics apart from $\mathbf{CK}^{\#}$ undercuts Peano arithmetic even in the extensional fragment.

However, we can encode disjunctive syllogism into our formulae in a rather interesting way, due to Meyer [89,90], and thereby have a presentation of classical arithmetic within our systems that serves a number of useful functions.

LEMMA 11.26 *In any of our logics* $\mathbf{X}$*, we have*

$$A \vee f,\ \sim A \vee B \vee f\ \vdash\ B \vee f$$
$$\sim A \vee B \vee f,\ B \vee \sim B\ \vdash\ A \wedge t \to B \vee f$$

Proof: Firstly, $A \vee f \vdash t \to (A \wedge t) \vee f$ by simple t-moves. Factoring a $\sim B$ in on the antecedent and consequent gives

$$A \vee f \vdash \sim B \wedge t \to (\sim B \wedge A \wedge t) \vee (\sim B \wedge f) \qquad (\alpha)$$

Similarly, $\sim A \vee B \vee f \vdash t \to \sim A \vee B \vee f$, and contraposing gives $\sim A \vee B \vee f \vdash \sim B \wedge A \wedge t \to f$ So, $\sim B \wedge A \wedge t \to f$ factored into the antecedent of (α) gives

$$\begin{aligned} A \vee f, \sim A \vee B \vee f &\vdash \sim B \wedge t \to f \vee (\sim B \wedge f) \\ &\vdash \sim B \wedge t \to f \\ &\vdash t \to B \vee f \\ &\vdash B \vee f \end{aligned}$$

as we wished to show.

For the second half of the result, $\sim A \vee B \vee f \vdash A \wedge \sim B \wedge t \to f$ by t-introduction, contraposition and de Morgan identities, so we get

$$\sim A \vee B \vee f \vdash A \wedge \sim B \wedge t \to B \vee f$$

Simple manipulations show that $\vdash A \wedge B \wedge t \to B \vee f$, so $\sim A \vee B \vee f \vdash A \wedge (B \vee \sim B) \wedge t \to B \vee f$. It follows that $\sim A \vee B \vee f,\ B \vee \sim B \vdash A \wedge t \to B \vee f$ as we desired. ◁

THEOREM 11.27 *If A is extensional and A is provable in classical Peano arithmetic, then $\vdash A \vee f$ in $\mathbf{X}^{\#}$ for any of our logics $\mathbf{X}$.*

Proof: Consider a Hilbert-style proof of A from the standard axiomatisation of Peano arithmetic, in an arrow-free form (translate all instances of $A \to B$ to $\sim A \vee B$). In the proof, disjoin all lines with an f. This new proof will be valid in $\mathbf{X}^{\#}$. Firstly, any logical axiom in classical Peano arithmetic is provable in $\mathbf{X}^{\#}$, because excluded middle is provable in $\mathbf{X}^{\#}$ (for extensional formulas) and so, all theorems of classical logic are provable in $\mathbf{X}$. For arithmetic axioms, those without connectives are identical in both systems. For those with, if $A \supset B$ is an axiom in Peano arithmetic (perhaps with quantifiers) then $A \to B$ is an axiom in $\mathbf{X}^{\#}$, and since $A \vee \sim A$ is provable (A being extensional) $\sim A \vee B$ is provable in $\mathbf{X}^{\#}$ too. Secondly, we must show that the rules are valid in the new proof. Here, for *modus ponens* for the material conditional, if our new proof has $A \vee f$ and $\sim A \vee B \vee f$, we can validly deduce $B \vee f$, as our lemma shows. If the rule is induction, our new proof will have $A(\underline{0}) \vee f$ and $\forall x\big(\sim A(x) \vee A(x')\big) \vee f$. We can also prove $A(x') \vee \sim A(x)$, so by our lemma we have $\forall x\big(A(x) \wedge t \to A(x') \vee f\big)$, but this easily gives $\forall x\big((A(x) \wedge t) \vee f \to (A(x') \wedge t) \vee f\big)$. This and $(A(\underline{0}) \wedge t) \vee f$ are enough to get the induction going in $\mathbf{X}^{\#}$, to prove $\forall x\big(A(x) \vee f\big)$ and hence $\forall x A(x) \vee f$ as desired. So, the proof is valid in $\mathbf{X}^{\#}$ as desired. ◁

11.3 Gödel's Theorem

Given the results of the previous section, it is simple to reproduce Gödel's incompleteness results in our arithmetics, following Meyer [90]. (Of course, as they are subsystems of Peano arithmetic, we already know that they are incomplete. Nonetheless, it is interesting to see how the proof proceeds.)

Definition 11.5 Gödel's coding can be recast within our arithmetics. Specifically:

- There is a recursive bijection between formulae and numerals, such that for each $\underline{n}$, $\mathrm{frm}(\underline{n})$ is a formula, for each A, $\ulcorner A \urcorner$ is a numeral, $\mathrm{frm}(\ulcorner A \urcorner) = A$ and $\ulcorner \mathrm{frm}(\underline{n}) \urcorner = \underline{n}$.
- For each term t, the formula $\mathrm{frm}(\underline{n})t$ is given by replacing any free occurrences of x in $\mathrm{frm}(\underline{n})$ by t.
- Define the set $r(n)$ by setting $m \in r(n)$ iff $\vdash \mathrm{frm}(\underline{n})\underline{m}$. (Clearly, this is relative to a formal arithmetic.) This is said to be the set *weakly represented by* $\mathrm{frm}(\underline{n})$.
- S is *weakly representable in* $\mathbf{X}^{\#}$ iff there is some n such that $S = r(n)$ in $\mathbf{X}^{\#}$.

THEOREM 11.28 *For each* X, S *is weakly representable in* $\mathbf{X}^{\#}$ *iff* S *is recursively enumerable.*

Proof: Clearly the class of provable sentences in $\mathbf{X}^{\#}$ is recursively enumerable. So, for each n we can provide a recursive enumeration of $r(n)$.

Conversely, we have the result for Peano arithmetic. Let $S = r(n)$ in Peano arithmetic. Then as $\mathrm{frm}(\underline{n})\underline{m}$ is provable in Peano arithmetic iff $\mathrm{frm}(\underline{n})\underline{m} \vee f$ is provable in $\mathbf{X}^{\#}$, $S = r(\ulcorner \mathrm{frm}(\underline{n}) \vee f \urcorner)$. ◁

In the previous theorem, we may note that every recursively enumerable set is weakly represented by an *extensional* formula A with only x free.

THEOREM 11.29 $\mathbf{X}^{\#}$ *is incomplete.*

Proof: Let $\mathfrak{G}$ be the set of all n such that $n \in r(n)$. In other words

$$n \in \mathfrak{G} \text{ iff } \vdash \mathrm{frm}(\underline{n})\underline{n} \qquad (\alpha)$$

Clearly $\mathfrak{G}$ is recursively enumerable, so it is weakly represented by some formula G. Let $\ulcorner G \urcorner = \underline{g}$. Then

$$n \in \mathfrak{G} \text{ iff } \vdash \mathrm{frm}(\underline{g})\underline{g} \qquad (\beta)$$

Consider $\sim G$. Specifically, $\ulcorner{\sim G}\urcorner \in \mathfrak{G}$ iff $\vdash G\ulcorner{\sim G}\urcorner$, by (β), and $\ulcorner{\sim G}\urcorner \in \mathfrak{G}$ iff $\vdash {\sim G}\ulcorner{\sim G}\urcorner$, by (α). In other words, $\vdash G\ulcorner{\sim G}\urcorner$ iff $\vdash {\sim G}\ulcorner{\sim G}\urcorner$. However, $\mathbf{X}^{\#}$ is negation consistent, so neither $G\ulcorner{\sim G}\urcorner$ nor ${\sim G}\ulcorner{\sim G}\urcorner$ is provable. $\triangleleft$

11.4 Strong Representability

So, we have a way of weakly representing all recursively enumerable predicates in our arithmetics, and this provides a Gödel style incompleteness proof. However, *weak* representability is quite weak. You need much more to prove the diagonal lemma. Specifically, you need a notion of *strong representability*.

Definition 11.6 A formula $A(x)$ with one free variable x *very strongly represents* the set S of natural numbers just when:

$$\vdash \forall x\big(A(x) \leftrightarrow x = \underline{n}\big) \text{ if } n \in S$$

This is quite strong, in that it requires that the biconditional be a relevant one. A slightly weaker notion is possible: A formula $A(x)$ *strongly represents* the set S just when:

$$\vdash \forall x\big(A(x) \equiv x = \underline{n}\big) \text{ if } n \in S$$

where $A \equiv B$ is defined as $(A \supset B) \wedge (B \supset A)$. These definitions can be expanded in natural ways to define the representability of functions from N^n to N as well. The function f from N^2 to N is *weakly represented* by A just when

$$\vdash A(\underline{m}_1, \underline{m}_2, \underline{n}) \text{ iff } f(m_1, m_2) = n$$

it is *strongly represented* by A just when

$$\vdash \forall x\big(A(\underline{m}_1, \underline{m}_2, x) \equiv x = \underline{n}\big) \text{ if } f(m_1, m_2) = n$$

and it is *very strongly represented* by A just when

$$\vdash \forall x\big(A(\underline{m}_1, \underline{m}_2, x) \leftrightarrow x = \underline{n}\big) \text{ if } f(m_1, m_2) = n$$

Clearly, if a set is very strongly representable, it is strongly representable, and if it is strongly representable, it is weakly representable. In classical Peano arithmetic, these three notions coincide. In our arithmetics, this is an open question. Here's an example that illustrates why.

In classical arithmetic, each recursive function is very strongly representable. One recursive function is the identity function $f_=$.

$$f_=(x,y) = \begin{cases} 1, & \text{if } x = y; \\ 0, & \text{otherwise.} \end{cases}$$

It is very strongly represented by the predicate $(x = y \wedge z = \underline{1}) \vee (x \neq y \wedge z = \underline{0})$. The reasoning is simple. If $m = n$, then $z = 1 \to (\underline{m} = \underline{n} \wedge z = 1)$, so $z = 1 \to (\underline{m} = \underline{n} \wedge z = \underline{1}) \vee (\underline{m} \neq \underline{n} \wedge z = \underline{0})$. Conversely, $(\underline{m} = \underline{n} \wedge z = \underline{1}) \to z = \underline{1}$, and as $\underline{m} \neq \underline{n}$ implies *anything* classically speaking, $(\underline{m} \neq \underline{n} \wedge z = \underline{0}) \to z = \underline{1}$ too. So, we have the converse. The case for $m \neq n$ is similar. Clearly in logics like **R** and below, both directions are problematic. The first, from $z = \underline{1}$ to deduce $\underline{m} = \underline{n} \wedge z = \underline{1}$ can be fixed by using an enthymematic implication (allowing the importation of irrelevant truths like $\underline{m} = \underline{n}$), but the second isn't so easily fixed. The false proposition $\underline{m} \neq \underline{n}$ does not lead to $z = \underline{1}$, no matter how hard you try. For the details, we need an excursus into the finite models of arithmetic of Meyer and Mortensen. If the reader does not want those details, take the result on trust, and skip to the end of the excursus.

EXCURSUS: We need just consider one model of $\mathbf{R}^{\#}$. The model is a simple relational structure of the kind discussed in Chapter 4. Its domain is the set $\{0, 1\}$, and the functions of successor, addition and multiplication are given by the operations modulo 2. So, $0' = 1$, $1 + 1 = 0$, and $1 \times 0 = 0$, for example. This does not yet make a relational structure, because we have not learned what propositional structure we need to evaluate the truth values of formulae, and we do not know how to interpret identity in this structure. This needs a little subtlety. For even though the addition and multiplication axioms are satisfied by these functions, it is harder to ensure that $\underline{0} \neq x'$ always comes out as true, because manifestly, we do have $0 = 1'$. Hence the subtlety: If the logic permits it, we can have *both* $0 = 1'$ *and* $0 \neq 1'$. And a logic that permits this is given by the propositional structure **RM3**. (To get **RM3**, just take the structure of **BN4**, and restrict yourself to evaluations that contain only t, f and b. Propositions can be 'true', 'false' or 'both', but not 'neither'.) This structure is well known to be a model of the relevant logic **R**. And in this structure, a proposition which is evaluated as b is designated, and its negation, also b is also designated. For identity, we take it that $0 = 1$ and $1 = 0$ take the value f, and that $0 = 0$ and $1 = 1$ take the value b. This makes sense on our interpretation. For even though $0 = 0$ is clearly true, we have said that 0 represents each even number, so $0 = 0$ is false too, because $4 = 3088$ is not true, for example. It is simple but tedious to show that in this relational structure, each Peano axiom comes out as at least *both*. So, we have a model of $\mathbf{R}^{\#}$. Now we can use it to demonstrate our result.

Now, in this model

$$(\underline{1} = \underline{0} \wedge z = \underline{1}) \vee (\underline{1} \neq \underline{0} \wedge z = \underline{0}) \to z = \underline{1}$$

fails in the case where $z = \underline{0}$, because the antecedent is b and the consequent f. □

So $f_=$ is not strongly represented by $(x = y \wedge z = \underline{1}) \vee (x \neq y \wedge z = \underline{0})$ in arithmetics no stronger than $\mathbf{R}^{\#}$. Whether it is strongly representable (by something else) in these arithmetics is an open question.

It may be thought that we could avoid this problem by can simply *extending* our language to incorporate function symbols for all of the recursive functions. For example, we could add the function $f_=$ satisfying the following definition.

$$\forall x \forall y \big(f_=(x, y) = \underline{1} \leftrightarrow x = y\big)$$
$$\forall x \forall y \big(f_=(x, y) = \underline{0} \leftrightarrow x \neq y\big)$$

But doing this would not conservatively extend the arithmetics. Given such a function $f_=$ we can prove $\underline{0} = a' \to \underline{0} = \underline{1}$, which is not a theorem of any arithmetic $\mathbf{R}^{\#}$ or weaker (for example, $\underline{0} = \underline{2} \to \underline{0} = \underline{1}$ fails the two-element model). Here is the proof. By both axioms for $f_=$ we have

$$\vdash f_=(\underline{0}, a') = 1 \leftrightarrow \underline{0} = a' \qquad \vdash f_=(\underline{0}, a') = \underline{0} \leftrightarrow \underline{0} \neq a'$$

Now $\vdash \underline{0} \neq a'$, so we have $\vdash f_=(\underline{0}, a') = \underline{0}$, which gives $\vdash \underline{0} = \underline{1} \leftrightarrow \underline{0} = a'$, by substitution.

So, there are clearly a number of important issues in representing sets in our arithmetics. We leave these issues for further study.

11.5 The ω Rule

Peano arithmetic is an interesting theory about the natural numbers, but it is an incomplete one. Given Gödel's incompleteness results, we cannot take the story of any arithmetic $\mathbf{X}^{\#}$ to be the whole story about numbers. The existence of models of $\mathbf{X}^{\#}$ which do not validate everything in the standard model means that $\mathbf{X}^{\#}$ does not 'pin down' arithmetic as accurately as we would like. We have shown (using Meyer and Friedman's result) that arithmetics no stronger than $\mathbf{R}^{\#}$ are even worse than $\mathbf{P}^{\#}$ at pinning down arithmetic truths. This is not a problem in the logic — it's the general problem of recursively axiomatising the truths in the standard model (this is independent of the logic of the underlying conditional — the extensional truths in the standard model are independent of the underlying logic). Gödel has shown that there is no hope of giving a complete recursive axiomatisation of arithmetic truth however we do it. This doesn't mean that there is *no* axiomatisation of arithmetic truth, provided that we are liberal enough in our conception of axiomatisation. It only follows that any axiomatisation is not going to be recursive. It is simple to show, for example, that the standard model validates the rule

$$A(\underline{0}), A(\underline{1}), A(\underline{2}), \ldots \vdash \forall x A(x)$$

We call it the ω-rule. Adding this to the axiomatisation of $\mathbf{X}^{\#}$ (and removing PMI, which follows from the ω-rule) results in the arithmetic $\mathbf{X}^{\#\#}$. We can show that any arithmetic $\mathbf{X}^{\#\#}$ truly models arithmetic truth in the following sense.

THEOREM 11.30 *For any arithmetic $\mathbf{X}^{\#\#}$, if A is an extensional sentence true in the standard model, then A is a theorem of $\mathbf{X}^{\#\#}$.*

Proof: By induction on complexity of A, when put into a normal form in which the only negations apply to atomic formulae. (By de Morgan identities (over conjunction, disjunction and the quantifiers) and double negation any formula is provably equivalent to a formula of this form.) If A is atomic, then the result holds by Theorem 11.25. $\mathbf{X}^{\#}$, and hence $\mathbf{X}^{\#\#}$, prove all true numerical equations and inequations. The induction steps for conjunction, disjunction and existential quantification are trivial, and universal quantification is given by the ω rule. ◁

11.6 Truth and Triviality

Arithmetic in $\mathbf{CK}^{\#}$ and $\mathbf{CK}^{\#\#}$ is interesting. The results of earlier sections show that in all respects $\mathbf{CK}^{\#}$ is like $\mathbf{P}^{\#}$. The axioms of arithmetic suffice to prove all instances of excluded middle in its language. However, once we add a T-predicate that satisfies the biconditional

$$\vdash T\ulcorner A\urcorner \leftrightarrow A$$

we no longer have the full strength of classical logic — excluded middle may not extend to formulae involving T. Call the system $\mathbf{X}^{\#}$ with a T-predicate $\mathbf{X}^{\#}T$. So, the arithmetic is classical with respect to T-free sentences, and any sentences whose truth-value is *grounded* (in Kripke's sense [63]) in T-free sentences (like $T\ulcorner \underline{2}+\underline{2}=\underline{5}\urcorner$ or $T\ulcorner\forall x(x+y=y+x)\wedge T\ulcorner\underline{2}+\underline{2}=\underline{5}\urcorner\urcorner$). However, in things like the liar sentence L which satisfies

$$\vdash \sim T\ulcorner L\urcorner \leftrightarrow L$$

the classical results like excluded middle could fail. We have an interesting way of getting past the devastating results of Tarski's "Theorem" which tells us that there's no way to consistently add to arithmetic a predicate of whose extension contains the (Gödel numbers of) true statements of that language (and only those). Similarly with Löb's Theorem. The way is open to introduce a real provability predicate Pr which satisfies $\vdash Pr\ulcorner A\urcorner \rightarrow A$. The way ahead from a contraction-free perspective is to reject contraction for these propositions. Then the usual proof to impossibility is lost, but the damage in the weakening of the logic is limited to only those statements that are not grounded. As we've seen, the T-free sentences in $\mathbf{CK}^{\#\#T}$ are classical, so it follows that $T\ulcorner A\urcorner$ is classical if A is, and so on.

However, this is not enough to guarantee us that $\mathbf{CK}^{\#\#T}$ is worthwhile, because even though the usual argument in Löb's theorem will fail, other triviality arguments may not. And this is what happened in the slightly stronger arithmetic $\mathbf{Ł}_{\infty}^{\#\#T}$, as we will show.

$\mathbf{Ł}_{\infty}^{\#\#T}$ is an interesting system only insofar as it could give us a non-triviality proof for $\mathbf{CK}^{\#\#T}$. A model for $\mathbf{Ł}_{\infty}^{\#\#T}$ is a structure with natural numbers as domain, and in which the Peano axioms are true, with the T-scheme added, in Łukasiewicz's infinitely valued logic. The logic for $\mathbf{Ł}_{\infty}^{\#\#T}$ is $\mathbf{Ł}_{\infty}$, which is useful for non-triviality arguments because we can use methods from analysis to get nice results about the interval $[0,1]$. It was with this in mind that I studied models of $\mathbf{Ł}_{\infty}^{\#\#T}$. Unfortunately, I found a triviality argument, rather than a non-triviality argument.

Firstly, we need enough self-reference to concoct the offending formulae. For this, we need a special form of the diagonalisation lemma that allows for formulae with free variables to be diagonalised. Recall the diagonalisation function, which takes Gödel numbers of formulae and returns the Gödel number of its diagonalisation, given a particular free variable we've picked out for the purpose. Let the variable in our case be x, then the diagonalisation of A is $\exists x(x = \ulcorner A \urcorner \wedge A)$.

LEMMA 11.31 *If we have an arithmetic theory that is classical in the arithmetic fragment, and which represents the diagonal function* $diag$*, then for any formula* $B(y)$ *with at least the variable* y *free, there is a formula* R *(with at most the variables other than* y *free in* $B(y)$ *free in it) where* $\vdash R \leftrightarrow B(\ulcorner R \urcorner)$ *in that arithmetic.*

Proof: The proof is standard. Let $A(x, y)$ represent $diag$ in the arithmetic. So, for any n, k if $diag(n) = k$ then $\vdash \forall y(A(\underline{n}, y) \leftrightarrow y = \underline{k})$.

Let F be the formula $\exists y(A(x, y) \wedge B(y))$. F is a formula with at most the variables other than y free in $B(y)$ free in it. Let $n = \ulcorner F \urcorner$ and let R be the expression $\exists x(x = \underline{n} \wedge \exists y(A(x, y) \wedge B(y)))$. As $\underline{n} = \ulcorner F \urcorner$, R is the diagonalisation of F, and it has the same variables free in it as in F, except for x. So we must have $\vdash R \leftrightarrow \exists y(A(\underline{n}, y) \wedge B(y))$. Let $\underline{k} = \ulcorner R \urcorner$; then $diag(n) = k$ and we must have $\vdash \forall y(A(\underline{n}, y) \leftrightarrow y = \underline{k})$ as A represents $diag$. It follows that $\vdash R \leftrightarrow \exists y(y = \underline{k} \wedge B(y))$, which gives $\vdash R \leftrightarrow B(\underline{k})$ as desired. $\triangleleft$

Now we can define our offending formulae. Let $A_0, A_1, A_2, \ldots$ be defined as follows:[1]

$$A_0 = {\sim}\forall x \exists y(R(x+1, y) \wedge Ty) \qquad A_{n+1} = A_0^{n+1}$$

where R is a recursive predicate defined to represent the Gödel codes of $A_0, A_1, A_2 \ldots$. This means that Rxy is true whenever y is the Gödel code of sentence number x in the list above, and false otherwise. Such a predicate R is expressible in the arithmetic fragment of the language. Consider how all of the formulae A_i are 'manufactured' from the previous ones by a very simple procedure. Given the code of A_i, the code of A_{i+1} is found by applying a simple transformation. This can be packed into a recursive definition. We don't yet have the code $\ulcorner R \urcorner$ of R, so we'll leave this as the argument place of a function. Then we can let $h(\ulcorner R \urcorner, \underline{0})$ be the Gödel number of A_0, as a function of whatever code we give R itself. Then, we can let $h(\ulcorner R \urcorner, \underline{n+1})$ be the Gödel number of A_{n+1}, given by a simple transformation on $h(\ulcorner R \urcorner, n)$. The resulting function is recursive. (We defined it by recursion in simple arithmetic functions dictating the way formulae are made up out of subformulae). So, this function h of two free variables can be represented by some predicate in classical arithmetic. Call this predicate H. To get the value of $\ulcorner R \urcorner$, we need it to satisfy:

$$\vdash Rxy \leftrightarrow H(\ulcorner R \urcorner, x, y)$$

But we have such an R by the diagonalisation lemma that we proved.

We need just one more lemma before we get to the proof of inconsistency.

LEMMA 11.32 *In any* $\mathbf{Ł}_\infty$ *model, if* $I(A) \neq 1$ *then for some* n*,* $I(A^n) = 0$*. This is called the iterated fusion fact.*

Proof: Let $I(A) = a$. Then $I(A^n) = \max(0, 1 - n(1-a))$, so if $n > 1/(1-a)$ then $I(A^n) = 0$. Thus, if $I(A) \neq 1$ then for some n, $I(A^n) = 0$. ◁

This fact is associated with the validity of *Hay's Rule* in $\mathbf{Ł}_\infty$ [57]. The rule is infinitary: If $\vdash {\sim} A^n \to A$ for each n, then $\vdash A$.

THEOREM 11.33 $\mathbf{Ł}_\infty^{\#\#\top}$ *is inconsistent.*

Proof: A_0 is designed to "say" 'not all of the A_i are true, for $i > 0$' Assuming this is how A_0 is to be interpreted, we could reason like this. If A_0 were true, then each A_i would be true too by construction. So, A_0 must be untrue. But then, some A_n would be positively false (iterated fusions do this), making A_0 true. A contradiction.

This means that something must go wrong with the interpretation of A_0, on pain of inconsistency. The universal quantifier must pick out more than just the standard numbers. The argument goes like this, formally.

Consider the value $I(A_0)$ of A_0 in some model. If $I(A_0) \neq 1$ then $I(A_{n+1}) = I(A_0^{n+1}) = 0$ for some n (by iterated fusion). This means that $I(T\ulcorner A_{n+1}\urcorner) = 0$ by the T-scheme, and hence $I\big(\exists y\big(R(\underline{n}{+}1, y) \wedge Ty\big)\big) = 0$ by the definition of R. Hence, $I(A_0) = 1$, contradicting what we assumed.

So, we must have $I(A_0) = 1$, and hence $I(A_0^{n+1}) = I(A_{n+1}) = 1$ for each n. This means that $I\big(\exists y\big(R(\underline{n}{+}1, y) \wedge Ty\big)\big) = 1$ for each n. In other words, $\vdash \exists y\big(R(\underline{n}{+}1, y) \wedge Ty\big)$ for each n, which by the ω rule gives $\vdash \forall x \exists y\big(R(x+1, y) \wedge Ty\big)$. However $\vdash A_0$ means that $\vdash {\sim} \forall x \exists y\big(R(x+1, y) \wedge Ty\big)$, which is contradicts what we have seen. So, the theory is inconsistent. ◁

Interpretation The triviality proof uses a subtle form of contraction. The problem is in $\mathbf{Ł}_\infty$ and its susceptibility to contraction as a result of infinite iterations. As you can see, Hay's rule (from each ${\sim} A^n \to A$ to A) is a generalisation of *reductio* — in the form that allows the inference from ${\sim} A \to A$ to A. The rule is valid in $\mathbf{Ł}_\infty$ because if A is not true, then the series A^n will eventually reach the false, and so one ${\sim} A^n \to A$ must come out as untrue (being eventually equivalent to A where A^n is the false). So conversely, if each ${\sim} A^n \to A$ is true, then we must have A true as well.

There are two possible responses to this proof. One is to ditch the ω-rule from the proof theory. This amounts to admitting non-standard models (and rejecting the standard model). Unfortunately, the only conceivable ground for doing this is some principled decision against infinitary rules — and $\mathbf{Ł}_\infty$ already has an infinitary rule of its own. If we are going to admit an infinitary rule as intelligible, then we had better admit ω, because it comes out as valid if our metatheoretic natural numbers coincide with our object-level natural numbers. And surely this is desirable.

The other response is to reject $\mathbf{Ł}_\infty$ as an adequate contraction-free logic. This seems much more plausible, as the offending inference is obviously contraction related. It dictates that for any untrue proposition, a finite number of fusions with itself is enough to bring it down to The False. This is similar to the situation in finitely valued logics, that we will discuss in Chapter 13. There the problem is because there is a fixed number of iterations to contraction. Here, the number is not fixed (it can be arbitrarily large) but it is finite. And as we have seen, this is enough to render the arithmetic trivial.

The arithmetics $\mathbf{CK}^{\top\#\#}$ and weaker are immune from this problem (Hay's rule is invalid in the underlying logics), and they hold out a hope for a decent non-trivial self-referential arithmetic. All indications show that arithmetics like these are ways to keep together our intuitions about truth, self-reference, and arithmetic. The devastating results of the liar, and Curry's paradox are not provable in $\mathbf{CK}^{\top\#\#}$ (at least in the standard way; we need a consistency proof to be really assured of this point.) Even Löb's result is dented in $\mathbf{CK}^{\top\#}$ even though the arithmetic remains classical.

EXCURSUS: This point is subtle. Löb's result is provable in classical Peano arithmetic, so it is also provable in $\mathbf{CK}^{\#}$. However, in an arithmetic like $\mathbf{CK}^{\top\#}$ another alternative is open to a theorist. Even though Löb has shown us that no arithmetically representable provability predicate Prv will satisfy $\vdash \mathsf{Prv}\ulcorner A \urcorner \rightarrow A$, clearly other provability predicates do. For example, if Prv is a classically given provability predicate, then Prv^*, which is given by the definition:

$$\mathsf{Prv}^*\, x =_{\mathrm{df}} (\mathsf{T}\, x \wedge \mathsf{Prv}\, x)$$

will clearly satisfy all of the conditions of a provability predicate (check them) and it is also clear that $\vdash \mathsf{Prv}^*\ulcorner A \urcorner \rightarrow A$ too. Löb's theorem is useful in that it shows us that provability is not really captured by the extensional means available to us in classical arithmetic. It doesn't show us that real provability (which entails truth) is out of the reach of formalisation. □

The way ahead involves more work on the deductive strength of the arithmetic, and hopefully, some kind of indication of its consistency. Both, however, must be left for another occasion.

11.7 Note

[1] It was an insight of Ben Robinson that pointed me in the direction of defining a series of formulas with this general structure. I'm very grateful to him for the idea.

Chapter 12
Naïve Comprehension

The March 25, 1974 issue of the German periodical DER SPIEGEL
features a cover photograph of a world-weary youngster
under the blaring headline, "Macht Mengenlehre krank?"
("Is set theory making our children sick?")

— STEPHEN POLLARD *Philosophical Introduction to Set Theory* [111]

Paradoxes don't just arise in the context of arithmetic and truth. Property theory and set theory are also shaped by the way to respond to the paradoxes in their field. We will look at ways to develop property and set theories in the context of our favourite logics.

12.1 Comprehension Defined

Consider properties. Given a formula $A(x)$ with one free variable x (in an interpreted first-order language) it is plausible to suppose that there is a property $\langle x : A(x)\rangle$ that an object a has if and only if $A(a)$ — that is, the property of 'being an A.' Furthermore, if $A(x, y_1, \ldots, y_n)$ is a formula with each of $x, y_1, \ldots, y_n$ free, it is plausibe to suppose that for each possible choice of denotation of $y_1, \ldots, y_n$ there is a property $\langle x : A(x, y_1, \ldots, y_n)\rangle$ too. (So, for example, given any object x, there is a property of *being an* x, even if our language does not have a name for the object x.)

EXCURSUS: This account of properties means that we take there to be properties like 'being green or red' and 'not being tall'. Some realists about properties are loath to admit disjunctive properties, negative properties and the like, among the pantheon of 'real' properties. Their project is to categorise a family of universals that count as the 'ontologically required' properties. Then all other less robust properties such as disjunctions and negations are expressed in terms of their more robust cousins. This project is not of concern to us. We can simply note that there is *something* that it is for a cup to be either red or green. Yes, its "redness or greenness" is dependent on either redness or greenness (and perhaps these properties are dependent on others) and this relation of property dependence between properties is interesting, but it does not follow that there is *nothing* that it is for a cup to be either red or green. Our account of the logical structure of properties is not concerned with differentiating between complex and simple properties. We admit them all without distinction. □

The natural way to formalise the claim that $\langle x : A(x)\rangle$ is the property of 'being an A' by endorsing the naïve comprehension scheme.

Definition 12.1 The *naïve comprehension scheme* in a language $\mathcal{L}$ is the class of all statements of the form $\forall y\big(y \in \langle x : A(x)\rangle \leftrightarrow A(y)\big)$ where $A(y)$ is a formula that may or may not have the variable y (and other variables) free.

So, the language $\mathcal{L}$ must possess a binary relation denoted '$\in$' between objects. It is interpreted as follows: '$x \in y$' is true if and only if y is a property and the object x in fact has y. In natural deduction form the comprehension scheme can be expressed by the two rules.

$$\frac{X \Vdash A(a)}{X \Vdash a \in \langle x : A(x)\rangle}\ \in\mathrm{I} \qquad\qquad \frac{X \Vdash a \in \langle x : A(x)\rangle}{X \Vdash A(a)}\ \in\mathrm{E}$$

Definition 12.2 The theory $\mathbf{X}^{\langle:\rangle}$ is the result of adding the comprehension scheme to the first-order logic $\mathbf{X}$ (adding the predicate '$\in$' to the language if it is not already there).

We have already seen that the paradoxes pose a problem for this view of property abstraction — if **W** is accepted as valid. Consider the property $H_B = \langle x : x \in x \to B\rangle$. Ask yourself, does H_B have the property H_B? You may reason like this

1	(1)	$H_B \in H_B$	A
1	(2)	$(H_B \in H_B) \to B$	$1 \in$E
$1;1$	(3)	B	1, 2 MP
1	(4)	B	3 **WI**
0	(5)	$(H_B \in H_B) \to B$	1, 4 CP
0	(6)	$H_B \in H_B$	$5 \in$I
0	(7)	B	5, 6 MP

if you were in the habit of using **WI** to reason in contexts like this. Contemporary property theorists realise the difficulties. If you take **WI** as valid, you need to reject the comprehension scheme in its generality. Michael Jubien writes

> . . . it seems to me that at least some predicates of English express properties and that in some sentences, predicates do attribute properties to entities denoted by their subjects. In such cases I am inclined to say the sentences express propositions that are either true or false. It also seems to me that when a predicate *does* express a property we *can* convert it into a suitable property-denoting noun phrase. Now anyone who holds these views faces a serious semantical challenge: either to say why it is that the predicate of our recent example [a version of the heterological paradox] does not express a property *or*, assuming it does, to say why the entire sentence either does not make an attribution or else makes one that lacks truth-value. A genuine response to this challenge must not be *ad hoc* . . . The theory of properties I favour incorporates no attempt to respond to this sort of semantical challenge. [60]

This is truly a thorny problem for any decent formal theory of properties. Myhill agrees.

> Gödel said to me more than once, "There never were any set-theoretic paradoxes, but the *property-theoretic* paradoxes are still unresolved". [101]

He goes on to say that any who wish to endorse "Frege's Principle" (our naïve comprehension scheme) must use a non-classical logic. I agree. Contraction-free logics provide us a way to formalise property comprehension without falling into triviality. Our starting point is a result due to Richard White [167].

THEOREM 12.1 $\mathbf{L}_\infty^{\langle:\rangle}$ *is consistent.*

His proof is rather technical and out of the scope of this work, so we will not repeat it here. Suffice to say that he produces a natural deduction-like system for $\mathbf{L}_\infty^{\langle:\rangle}$ in which he shows that $\perp$ cannot be proved, by way of a normalisation procedure for proofs. This result has a pleasing corollary.

COROLLARY $\mathbf{X}^{\langle:\rangle}$ *is consistent for each of our favourite logics* $\mathbf{X}$.

This means that each of our logics is an adequate basis for developing a property theory with naïve comprehension in the spirit of Frege. We can agree that all formulas with one free variable determine properties, without being committed to everything following. We can prove results like Russell's paradox in property theoretic form. If R is $\langle x : \sim x \in x\rangle$ then we have

$$\vdash R \in R \leftrightarrow \sim(R \in R)$$

We have Curry's paradox too. With H_B being $\langle x : x \in x \to B\rangle$ as before, we can prove that

$$\vdash H_B \in H_B \leftrightarrow (H_B \in H_B \to B)$$

But disaster does not strike. Without contraction-related moves we cannot deduce B from Curry's paradox alone, and we cannot deduce a contradiction or triviality from Russell's paradox. R and H_B are odd properties, to be sure, but the strictures of contraction-free reasoning ensures that their oddness does not infect the whole theory with triviality.

However, reasoning without contraction is good not just for property theory: Despite the comments of Myhill and Gödel, it is also useful for set or class theory. This is due to a confusion in current talk about sets and classes. Two different accounts of what it is to be a set vie for attention in the literature. One account takes a set to be a 'collection constructed from antecedently available objects'. This seems to be the notion intended by Myhill and Gödel. Call these things sets_1. As far as we can see, sets_1 are consistent and free of paradox, even when using classical logic. To be sure, there is a difficulty when talking about all sets_1: The collection of all sets_1 cannot be a set_1, on pain of contradicting the definition. Yet, given all sets_1, we have antecedently available objects to collect into another object — *prima facie* another set_1. However we attempt to iron out these difficulties, talk about sets_1 naturally motivates Zermelo-style set theories such as **ZF** or **ZFC**, varying with regard to what set-construction methods are allowed (Are there choice-sets? Can we perform replacement?) and how 'far' the construction extends (Are there inaccessible cardinals? Are there Mahlo cardinals?). This notion has been at the centre of much mathematical research over the last hundred years.

However, there is another account of what it is to be a set is available for formalisation. According to this notion, a set is simply the extension of a property. Call these things sets_2. A set_2 can be found by abstracting away the intensions of properties. Co-extensional properties represent the same set_2. If we like, we can take a set_2 to be an arbitrary element of the equivalence class of co-extensional properties. It is *this* notion which is beset by paradoxes, because it inherits the paradoxical nature of properties. This notion admits a universal set_2 (the extension of the property of self-identity, the property 'the true', and any other property shared by everything) and many other non-well-founded sets_2. It also admits the Russell set_2, and many other entities which have caused no end of trouble.

Because there are two notions in play, and because subscripts are annoying if used for too long, we will call sets_1 *sets* and sets_2 *classes*, keeping in mind that the classes are not simply the classes of a classical class theory — these classes can be members of other classes, and even members of themselves.

Note that given plausible assumptions, there is a relationship between sets and classes. For any set y, there is the property $\langle x : x \in y \rangle$ of being a member of y. So, the class corresponding to that property has as members only the elements of the set y. This means that every set y is represented by a class y' (the extension of $\langle x : x \in y \rangle$) with the same members. We may as well identify y and y', because there is no discernible difference between them. This means that all sets are classes. However, not all classes are sets.

Maddy has drawn the set/class distinction, and she argues that classical set/class theories do not get the distinction right. She says

> In our search for a realistic theory of sets and classes, we begin with two desiderata:
>
> (1) classes should be real, well-defined entities;
>
> (2) classes should be significantly different from sets.
>
> The central problem is that it is hard to satisfy both of these. Von Neumann, Morse, Kelley and Reinhardt concentrate on (1) and succeed in producing theories with classes as real, well-defined entities, but they run afoul of (2) because their classes look just like additional layers of sets ... On the other hand, concentrating on (2) leads to Cantor's nonactual, or ineffable proper classes ... [75]

She is right. The theories on offer do not do justice to proper classes. Maddy's own response is to develop a theory of proper classes like Kripke's theory of truth — using Kleene's strong three-valued logic. This is a good start, but it does no justice to conditionals. In the rest of this chapter we will consider how to develop a more satisfactory property and class theory using the contraction-free resources we have accumulated.

To define classes as representatives of co-extensive properties, we must have some kind of understanding of identity. For example, we may define the function 'ext_of' to satisfy conditions like

$$\forall z(z \in x \leftrightarrow z \in y) \dashv\vdash \text{ext\_of}(x) = \text{ext\_of}(y),$$

which says that any two properties are co-extensive if and only if they have the same extension. We also can specify

$$\text{ext\_of}(\text{ext\_of}(x)) = \text{ext\_of}(x),$$

which states that any extension is its own extension. This means that we take an extension to be a representative property in the class of co-extensional properties. Clearly, identity features in these conditions, so we ought to have some idea of its behaviour in the context of property theory.

12.2 Identity Revisited

If two objects share all the same properties, they are identical (as they share the property of being identical to the first). Conversely, if a and b are identical, they share all the same properties. So it seems that we could define identity in property theory as follows:

Definition 12.3 $\mathrm{Eq}(a, b)$ is shorthand for $(\forall x)(a \in x \leftrightarrow b \in x)$

LEMMA 12.2 *In any theory* $\mathbf{X}^{\langle:\rangle}$ *we have* $\vdash \mathrm{Eq}(a, a)$, $\vdash \mathrm{Eq}(a, b) \to \mathrm{Eq}(b, a)$, *and* $\mathrm{Eq}(a, b) \vdash \mathrm{Eq}(b, c) \to \mathrm{Eq}(a, c)$. *Furthermore, in* $\mathbf{TW}^{\langle:\rangle}$ *and in every stronger theory,* $\vdash \mathrm{Eq}(a, b) \to \big(\mathrm{Eq}(b, c) \to \mathrm{Eq}(a, c)\big)$

Proof: A simple matter of unpacking the definitions. ◁

That result makes Eq an equivalence relation on the domain. More important is the next result, which shows that Eq is a good candidate for an identity relation

THEOREM 12.3 *In any theory* $\mathbf{X}^{\langle:\rangle}$, $\vdash \mathrm{Eq}(a, b) \to (A(a) \leftrightarrow A(b))$.

Proof:

1	(1)	$\mathrm{Eq}(a, b)$	A
1	(2)	$(\forall x)(a \in x \leftrightarrow b \in x)$	1 Eqdf
1	(3)	$a \in \langle x : A(x)\rangle \leftrightarrow b \in \langle x : A(x)\rangle$	2 ∀I
0	(4)	$a \in \langle x : A(x)\rangle \leftrightarrow A(a)$	∈I, ∈E
0	(5)	$b \in \langle x : A(x)\rangle \leftrightarrow A(b)$	∈I, ∈E
1	(6)	$A(a) \leftrightarrow A(b)$	3, 4, 5 Trans ↔
0	(7)	$\mathrm{Eq}(a, b) \to (A(a) \leftrightarrow A(b))$	1, 6 CP

We call this fact *the indiscernibility of* Eq*s*. ◁

LEMMA 12.4 $\vdash_{\mathbf{X}^{\langle:\rangle}} \big(\mathrm{Eq}(a, b) \to (\mathrm{Eq}(a, b) \to B)\big) \to \big(\mathrm{Eq}(a, b) \to B\big)$ *in any logic* $\mathbf{X}$ *that contains permutation.*

Proof:

1	(1)	$\mathrm{Eq}(a, b)$	A
1	(2)	$\big(\mathrm{Eq}(a, b) \to (\mathrm{Eq}(a, b) \to B)\big) \to \big(\mathrm{Eq}(b, b) \to (\mathrm{Eq}(b, b) \to B)\big)$	1 Theorem 12.3
3	(3)	$\mathrm{Eq}(a, b) \to (\mathrm{Eq}(a, b) \to B)$	A
3; 1	(4)	$\mathrm{Eq}(b, b) \to (\mathrm{Eq}(b, b) \to B)$	2, 3 MP
0	(5)	$\mathrm{Eq}(b, b)$	Fact
0; (0; (3; 1))	(6)	B	4, 5 MP^2
((1; 3); 0); 0	(7)	B	Permutations
1; 3	(8)	B	7 0-right
3	(9)	$\mathrm{Eq}(a, b) \to B$	1, 8 CP
0	(10)	$(\mathrm{Eq}(a, b) \to (\mathrm{Eq}(a, b) \to B)) \to (\mathrm{Eq}(a, b) \to B)$	3, 9 CP

◁

COROLLARY *If* $\mathbf{X}$ *is a logic containing permutation, then* $\vdash_{\mathbf{X}^{\langle:\rangle}} \mathrm{Eq}(a, b) \to \mathrm{Eq}(a, b) \circ \mathrm{Eq}(a, b)$. *That is, statements of the form* $\mathrm{Eq}(a, b)$ *contract.*

This corollary is quite similar to the necessity of identity statements (in most quantified modal logics, $\vdash a = b \to \Box\, a = b$) and it is not surprising. If we take it that Eq expresses identity, then we would expect Eq statements to be necessarily true if they are true at all, for the following reason. Given the strong comprehension scheme that gives us a property for every formula with one free variable, and granting that our language contains a modal operator, we get the necessity of true identities almost for free. Indiscernibility of Eqs gives

$$\vdash \mathsf{Eq}(a,b) \to \big(\Box\,\mathsf{Eq}(a,a) \to \Box\,\mathsf{Eq}(a,b)\big)$$

We have to assume that permutation is valid to get $\vdash \Box\,\mathsf{Eq}(a,a) \to \big(\mathsf{Eq}(a,b) \to \Box\,\mathsf{Eq}(a,b)\big)$ and then $\vdash \Box\,\mathsf{Eq}(a,a)$ to get

$$\vdash \mathsf{Eq}(a,b) \to \Box\,\mathsf{Eq}(a,b)$$

Given all of these properties, Eq seems to be rather appealing as a candidate for identity. However, all is not well with Eq. It also seems to have some odd properties.

For those who are concerned with relevance, the behaviour of Eq is rather troubling. after all, indiscernibility gives us such odd behaviour as theorems like

$$\vdash \mathsf{Eq}(a,b) \to (p \to p)$$

where p bears no relationship to a or b. We have $\vdash \mathsf{Eq}(a,b) \to (a \in \langle x : p\rangle \to b \in \langle x : p\rangle)$, and hence $\vdash \mathsf{Eq}(a,b) \to (p \to p)$ as desired. (This relies on vacuous abstraction as written, but it isn't necessary. If you replace p by $p \vee (p \wedge x = x)$, then x is free in the new version, but clearly, the new proposition is equivalent to p.) Those with 'relevant' intuitions take it to be undesirable that my self identity relevantly imply that if Socrates is wise, Socrates is wise, for example. Yet, our theory seems to make such things theorems — even when the underlying logic is relevant. There are at least two replies that the relevantist can make.

The first is to argue that the comprehension scheme as it stands is too strong. It takes it that $a \in \langle x : A(x)\rangle$ is relevantly equivalent to $A(a)$ no matter what a and $A(x)$ are. So, $a \in \langle x : \text{Socrates is wise}\rangle$ is equivalent to 'Socrates is wise.' This seems an odd way of saying things — is there really a property that a has if and only if Socrates is wise? Where that conditional is a relevant one? Quite possibly not. The most we could endorse is that $\vdash a \in \langle x : A(x)\rangle \equiv A(a)$ (where '$\equiv$' is the enthymematic biconditional) because a having $\langle x : A(x)\rangle$ may not be enough to relevantly imply $A(a)$, but other true information may need to be imported. This seems true enough, but it won't do to rid us of the paradoxes. It merely weakens them to this form:

$$\vdash \mathsf{Eq}(a,a) \to (p \supset p)$$

which are just as unpalatable. My self identity has just as little to do with the truth of 'Socrates being wise enthymematically implying Socrates' wisdom' as the original conditional. To fully take advantage of this line of approach you must excise vacuous properties such as $\langle x : \text{Socrates is wise}\rangle$ from the range of quantification in the comprehension scheme and in the definition of Eq. This seems difficult to do in practice, and even at a theoretical level.

However, another approach is possible that retains some of the insights of the first attempt. Instead of denying the existence of such hokey properties as $\langle x : \text{Socrates is wise}\rangle$ it suffices to note that their presence makes claims such as $\mathrm{Eq}(a, b)$ much stronger than the identity claim $a = b$. For $\mathrm{Eq}(a, b)$ is a universally quantified claim that when written out as a conjunction (and awfully long one, it must be said) has conjuncts equivalent to $A \leftrightarrow A$ for each sentence A (from the vacuous abstractions). Eq is a strong claim in that it is relevant to everything. In logics like **EW** and above, $\vdash (t \to t) \to t$, so $\vdash \mathrm{Eq}(a, b) \to (t \to t)$ gives $\vdash \mathrm{Eq}(a, b) \to t$. This means that if $\mathrm{Eq}(a, b)$ is true, it is equivalent to t. And t relevantly implies anything true, so we shouldn't expect anything less of $\mathrm{Eq}(a, b)$. So if our intuitions about the relevant connections of $a = b$ are anywhere near true, then $\mathrm{Eq}(a, b)$ is not the same claim as $a = b$. However, $\mathrm{Eq}(a, b)$ is true in a world just when $a = b$ is true in that world and vice versa, in the sense that $a = b \dashv\vdash \mathrm{Eq}(a, b)$. Eq has the same extension as identity but it differs in terms of its intensional, relevant connections. The seeming 'paradox' should upset us no more than the theoremhood of $t \to (p \to p)$ when t seems irrelevant to $p \to p$.

This leaves it open for us to consider whether we can define genuine identity in terms of property abstraction. Take $\mathrm{Eq}^*(a, b)$ to mean $a \in \langle x : A(x)\rangle \leftrightarrow b \in \langle x : A(x)\rangle$ for all properties $\langle x : A(x)\rangle$ that a and b *relevantly* have, using Dunn's characterisation of a relevant property [34]. The difficulty here is the definition of a relevant property.

$$\big(\rho x A(x)\big)a =_{\mathrm{df}} \forall x\big(x = a \to A(x)\big)$$

You can see the problem. Relevant predication is defined in terms of identity, so it is difficult to use relevant predication to *define* identity. The most we can do is to get some interesting relationships between them.

So, Eq need not be logically equivalent to identity — we need not have $\vdash \mathrm{Eq}(a, b) \leftrightarrow a = b$ — but we do have $\mathrm{Eq}(a, b) \dashv\vdash a = b$, where '$=$' is our intended identity predicate. From Chapter 4 we know that the identity predicate must satisfy the proof theoretical conditions

$$\vdash a = a, \quad a = b \vdash A(a) \to A(b)$$

and the equivalent semantic conditions that in any relational structure $\mathfrak{A}$,

$$\mathfrak{A} \vDash \nu_0 = \nu_1[d_0, d_1] \text{ if and only if } d_0 = d_1.$$

That is, $\nu_0 = \nu_1$ is comes out true in a structure if and only if their denotations are identical (as one would suppose). This does *not* say that on all other occasions $\nu_0 = \nu_1$ comes out as false, and it does *not* say how true $\nu_0 = \nu_1$ is when their denotations are identical. From this definition it follows that if $=_1$ and $=_2$ are identity predicates, then $a =_1 b \dashv\vdash a =_2 b$. This means that all identity predicates are weakly logically equivalent. However, our logics discriminate more than weak logical equivalence. So, we have Eq weakly logically equivalent to all other identity predicates, but for the reasons we have seen, it may not have the relevance properties we expect identity to have.

There is another identity predicate which assigns values to identities in a particularly stark way. Recall 'boolean identity', written '$=_{\mathrm{bool}}$', and interpreted by the relation R defined as

$$R(d_0, d_1) = \begin{cases} T & \text{if } d_0 = d_1, \\ F & \text{otherwise,} \end{cases}$$

where T and F are the top and bottom elements respectively of the propositional structure. This predicate assigns the starkest values to identities. If they are true, they are as true as they can be. Otherwise, they are as false as possible. This results in the validities

$$\top \vdash a =_{\mathrm{bool}} b \vee a \neq_{\mathrm{bool}} b, \quad a =_{\mathrm{bool}} b \wedge a \neq_{\mathrm{bool}} b \vdash \bot.$$

Boolean identity was seen to be useful in Chapter 4. We used it to prove a the upward Lowenheim-Skölem Theorem. Boolean identity will also be useful here. Though it is no better than Eq at characterising identity to strong logical equivalence (in cases where we expect excluded middle to fail — such as in the presence of vagueness — we would not expect identities to be immune, but $=_{\mathrm{bool}}$ satisfies excluded middle always) it will be useful to have around, because of its expressive power.

So, our investigations of identity have not brought us a hard-and-fast characterisation of the identity relation. If anything, it has shown us that there are a number of issues to be sorted out. Instead of pursuing this further, we have seen enough to proceed to analyse ways of encoding extensionality.

12.3 Extensionality

The Usual Way First, let's consider one way people have thought to encode extensionality into a naïve property theory. This is by adding an extensionality axiom on top of the naïve comprehension scheme. This means that we eschew talk of properties in their generality, and we take the comprehension scheme to talk of classes alone. So, we change notation, writing '$\{x : A(x)\}$' for '$\langle x : A(x)\rangle$'. The changed theory we call $\mathbf{X}^{\{:\}}$ in line with the changed notation.

So, we add some kind of extensionality axiom to express the fact that we're talking about classes and not just properties. We need to say that classes with the same elements are the same classes.

The standard candidate for an axiom of extensionality (equivalent to the one proposed by White [167], for example) is:

$$\frac{X \Vdash (\forall x)(x \in a \leftrightarrow x \in b)}{X \Vdash \mathrm{Eq}(a, b)} \quad \mathrm{EqI}$$

which is provably equivalent to the sentence:

$$(\forall y)(\forall z)((\forall x)(x \in y \leftrightarrow x \in z) \to \mathrm{Eq}(y, z)) \qquad \text{Ext}$$

Many who write in class theory (such as White [167]) use a the term like Eq for identity. As we have seen, this is harmless in the case of classical logic, because it gets the extension of identity right, and classical logic only takes extensions into account. For more sensitive logics like our favourites, differing intensions are important. We will show how important they can be by showing that Ext cannot be consistently added to $\mathbf{CK}^{\{:\}}$.

THEOREM 12.5 $\vdash_{\mathbf{CK}^{\{:\}}+\mathbf{Ext}} \perp$

We will approach the proof of this theorem by a lemma.

LEMMA 12.6 *If* A *contains no free variables, then* $\vdash_{\mathbf{CK}^{\{:\}}+\mathbf{Ext}} \mathsf{Eq}(\{x : A\}, \{x : t\}) \leftrightarrow A$

Proof:

1	(1)	$\mathsf{Eq}(\{x : A\}, \{x : t\})$	A
1	(2)	$b \in \{x : A\} \leftrightarrow b \in \{x : t\}$	1 SI
1	(3)	$A \leftrightarrow t$	2 $\in$I, $\in$E
0	(4)	$\mathsf{Eq}(\{x : A\}, \{x : t\}) \rightarrow (A \leftrightarrow t)$	1, 3 CP
5	(5)	$A \leftrightarrow t$	A
5	(6)	$b \in \{x : A\} \leftrightarrow b \in \{x : t\}$	5 $\in$I, $\in$E
5	(7)	$(\forall y)(y \in \{x : A\} \leftrightarrow y \in \{x : t\})$	6 $\forall$I
0	(8)	Ext	A
0	(9)	$(\forall y)(y \in \{x : A\} \leftrightarrow y \in \{x : t\}) \rightarrow \mathsf{Eq}(\{x : A\}, \{x : t\})$	8 $\forall$E twice
0; 5	(10)	$\mathsf{Eq}(\{x : A\}, \{x : t\})$	7, 9 MP
0	(11)	$(A \leftrightarrow t) \rightarrow \mathsf{Eq}(\{x : A\}, \{x : t\})$	5, 10 CP
0	(12)	$(A \leftrightarrow t) \leftrightarrow \mathsf{Eq}(\{x : A\}, \{x : t\})$	4, 11 $\wedge$I
0	(13)	$\mathsf{Eq}(\{x : A\}, \{x : t\}) \leftrightarrow A$	12 Lemma 2.12

Note that only the last step of this proof fails in relevant logics like **R**. ◁

This yields the proof of the theorem quite quickly.

Proof: $\vdash_{\mathbf{CK}^{\{:\}}+\mathbf{Ext}} \mathsf{Eq}(\{x : A\}, \{x : t\}) \leftrightarrow A$, and $\vdash_{\mathbf{CK}^{\{:\}}+\mathbf{Ext}} \mathsf{Eq}(\{x : A\}, \{x : t\}) \rightarrow \mathsf{Eq}(\{x : A\}, \{x : t\}) \circ \mathsf{Eq}(\{x : A\}, \{x : t\})$, so $\vdash_{\mathbf{CK}^{\{:\}}+\mathbf{Ext}} A \rightarrow A \circ A$. Thus, contraction gives Curry's paradox, and so, $\perp$. ◁

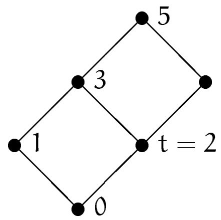
This is quite a disastrous result. Extentionality, in this form, yields triviality in the context of **CK**. In fact, it has some odd consequences in **C**.

THEOREM 12.7 $\vdash_{\mathbf{C}^{\{:\}}+\mathbf{Ext}} (A \leftrightarrow B) \rightarrow (A \leftrightarrow B) \circ (A \leftrightarrow B)$.

Proof: Given what has gone before, we have $\vdash_{\mathbf{C}^{\{:\}}+\mathbf{Ext}} \mathsf{Eq}(a, b) \rightarrow \mathsf{Eq}(a, b) \circ \mathsf{Eq}(a, b)$. And, it is easy to show that $\vdash_{\mathbf{C}^{\{:\}}+\mathbf{Ext}} \mathsf{Eq}(\{x : A\}, \{x : B\}) \leftrightarrow (A \leftrightarrow B)$. This gives us the result. ◁

Now it may be thought that in **C**, the contraction of all biconditionals spells disaster — that *everything* would contract. This would be the case if every proposition was provably equivalent to some biconditional, as it is the case in **CK**. But here, this doesn't work. The equivalence $\vdash A \leftrightarrow (A \leftrightarrow t)$ is restricted to logics with weakening, and it doesn't hold in **C**. In fact, a matrix for **C** (provided by `MaGIC` [154]) shows us that in

C, it is not the case that every proposition is provably equivalent to a biconditional.



a	$\sim a$
0	5
1	4
2	3
3	2
4	1
5	0

$\to$	0	1	2	3	4	5
0	5	5	5	5	5	5
1	0	2	0	4	1	5
2	0	1	2	3	4	5
3	0	0	0	2	0	5
4	0	0	0	1	2	5
5	0	0	0	0	0	5

$\circ$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	4	1	5	3	5
2	0	1	2	3	4	5
3	0	5	3	5	5	5
4	0	3	4	5	5	5
5	0	5	5	5	5	5

In this matrix it is easy to show that $(p \leftrightarrow q) \to (p \leftrightarrow q) \circ (p \leftrightarrow q)$ holds, but that $p \to p \circ p$ is falsified by setting $p = 1$. So, in this matrix all biconditionals contract, but some values do not. Therefore, not every proposition is equivalent to a biconditional. It follows that if *this* is the only damage that extensionality does in **C**, then we do not have a collapse into every instance of contraction.

Even though this does not give us a triviality proof of $\mathbf{C}^{\{:\}}$, the theory certainly isn't pretty. It seems that our class theory is teaching our logic what inferences are valid, for we get new *logical* truths (such as $(p \leftrightarrow q) \to (p \leftrightarrow q) \circ (p \leftrightarrow q)$) by way of facts about classes. Now there seems to be no real reason to hold that these extra propositions are true, independently of the result we've proven. So I take the fact that naïve class theory *doesn't* conservatively extend **C** to be a point against such a class theory in this logic. Whether or not there is a triviality proof, such as the one for **CK** remains open. It can be shown that the addition of $(p \leftrightarrow q) \to (p \leftrightarrow q) \circ (p \leftrightarrow q)$ to **C** does not yield the law of the excluded middle, so the **CX**-collapse [147] may not befall the class theory $\mathbf{C}^{\{:\}}$, but this does not mean that other triviality proofs are unavailable.

Although this kind of extensionality has been proposed by many, the triviality argument in **CK** and the considerations in the previous section about the excessive strength of Eq as an identity predicate indicates that this is perhaps the wrong tack to take. The way ahead is to take the original defining ideas of classes more seriously. Instead of abandoning property theory for class theory in the first instance, consider embedding class theory in property theory by defining classes as representatives in the collections of co-extensional properties.

A Better Way Let's model this interpretation of the relationship between properties and classes within the semantic structures of our theories. Consider a relational structure $\mathfrak{N} = \langle N; \mathfrak{P}, \varepsilon, \iota \rangle$ modelling $\mathbf{X}^{\langle:\rangle}$. It has a domain N of objects, at least some of which are properties. It as a map $\varepsilon : N \times N \to P$ (where P is the domain of the associated propositional structure $\mathfrak{P}$) which interprets the relation '$\in$'. For each $d_0, d_1 \in N$, $\varepsilon(d_0, d_1)$ is the value of '$v_0 \in v_1$' when v_0 denotes d_0 and v_1 denotes d_1. We write this value as '$d_0 \, \varepsilon \, d_1$' for short. There is also a map $\iota : N \times N \to P$ representing identity.

In this model, two domain elements d_0 and d_1 are co-extensional just when for each $e \in N$, $e \, \varepsilon \, d_0 = e \, \varepsilon \, d_1$. Co-extensionality is an equivalence relation on N, so we can pick representatives from each equivalence class (by choice). These representatives play the part of classes — they are merely arbitrarily picked representatives from among the co-extensive properties. So, the function ext_of can be be interpreted by the function

$c : N \to N$ that takes a domain element and returns the representative of its equivalence class. This motivates the following definition.

Definition 12.4 A relational structure $\mathfrak{N} = \langle N; \mathfrak{P}, \varepsilon, \iota, c\rangle$ is a *property-class* structure if and only if $c : N \to N$ is a map such that

- For each $d_0, d_1 \in N$, for each $e \in N$ $e \,\varepsilon\, d_0 = e \,\varepsilon\, d_1$ if and only if $c(d_0) = c(d_1)$
- For each $d, e \in N$, $e \,\varepsilon\, d = e \,\varepsilon\, c(d)$.

In each property-class model $\mathfrak{N}$ the class $C \subseteq N$ of elements in the range of c is said to be the class of *classes* in $\mathfrak{N}$.

The theory of these models, in the logic $\mathbf{X}$ is denoted $\mathbf{X}^{\langle:\rangle c}$. The function c interprets the function 'ext_of' in the language.

In each property-class structure

$$\mathsf{ext\_of}(\mathsf{ext\_of}(x)) = \mathsf{ext\_of}(x)$$

is valid. A representative of a class of co-extensive classes is itself in that class of co-extensive classes. So, it is its own representative. We also have the rule

$$\forall z(z \in \nu_0 \leftrightarrow z \in \nu_1) \dashv\vdash \mathsf{ext\_of}(\nu_0) = \mathsf{ext\_of}(\nu_1).$$

If d_0 and d_1 are in the same equivalence class in a model $\mathfrak{N}$ then $\mathfrak{N} \vDash \forall z(z \in \nu_0 \leftrightarrow z \in \nu_1)[d_0, d_1]$, because $\mathfrak{N}(z \in \nu_0)[d_0]$ and $\mathfrak{N}(z \in \nu_1)[d_1]$ are identical. This means that $\mathfrak{N}^t(\mathsf{ext\_of}(\nu_0)) = \mathfrak{N}^t(\mathsf{ext\_of}(\nu_1))$ and hence $\mathfrak{N} \vDash \mathsf{ext\_of}(\nu_0) = \mathsf{ext\_of}(\nu_1)[d_0, d_1]$ (for any identity predicate '$=$') as we desired. The converse is just as simple.

Because of this, we take $\{x : A\}$ to be shorthand for $\mathsf{ext\_of}\langle x : A\rangle$. This then gives us the following result.

THEOREM 12.8 *In any theory* $\mathbf{X}^{\langle:\rangle c}$, $\forall x(A \leftrightarrow B) \dashv\vdash \{x : A\} = \{x : B\}$

Proof: A simple corollary of the behaviour of the ext_of function. ◁

These are semantically driven clauses defining the concept of extensionality as arising from our initial conceptions of the relationhip between properties and sets. Note that this results in a weaker extensionality axiom that is often considered. We do not get

$$\vdash \forall x(A \leftrightarrow B) \to \{x : A\} = \{x : B\}$$

because we have no precise grip on the behaviour of identity.

If we interpret '$=$' by Eq, then this strong form of extensionality is trouble in $\mathbf{CK}^{\langle:\rangle c}$, because a simple modification of Theorem 12.5 is possible. The axiom is not likely under interpreting identity as boolean, either. When ν_0 and ν_1 denote co-extensional things $\mathsf{ext\_of}(\nu_0) =_{\mathsf{bool}} \mathsf{ext\_of}(\nu_1)$ will come out as T and the conditional trivially satisfied. Otherwise the consequent is F. To make the conditional true in this case we must have $\forall z(z \in \nu_0 \leftrightarrow z \in \nu_1)$ as F too. But there seems to be no assurance that this need be the case. It is not obvious that for non-co-extensional properties it is *absurd* to claim that they are co-extensional. So, our semantic conditions seem to validate the weaker axioms we have before us.

Once we have this distinction between properties and classes (which are the properties in the range of the function 'ext_of') we can embed class theory within property theory in a simple and natural way.

Definition 12.5 Let $\mathrm{Cl}(x)$ be shorthand for $\exists v_j\big(x =_{\mathsf{bool}} \mathsf{ext\_of}(v_j)\big)$, where v_j is free for x in $x =_{\mathsf{bool}} \mathsf{ext\_of}(v_j)$.

The idea is to restrict our quantifiers using the predicate Cl, giving us the ability to quantify over classes alone. However, this is a problematic procedure in non-classical logics in general, because restricted quantification is quite sensitive to the logic in which it is formulated. The problem is as follows. Consider the traditional definition of restricted universal quantification. If we wish to restrict quantification to objects satisfying a predicate P, we quantify as follows:

$$\forall x\big(P(x) \to A(x)\big)$$

where $A(x)$ is the formula to be quantified. This formula is intended to express the claim that all Ps are As. But this works in general only if the conditional has a number of desirable properties. Firstly, if we wish to deduce $A(a)$ from the fact that $P(a)$ and the universally quantified claim, we must have *modus ponens* for the conditional. This is fine for the conditional in each of our favourite logics, but it rules out the material 'conditional' because in none of our logics do we have $A, A \supset B \vdash B$. The second problem is this: If we have shown that *every* x satisfies $A(x)$, it seems to follow that all Ps are As too. We would like to have

$$\forall x A(x) \vdash \forall x\big(P(x) \to A(x)\big)$$

but this is only a realistic expectation in general when we have

$$A \vdash B \to A$$

which is valid in only **CK** of all our favourite logics. The final problem is this. It is plausible to suppose that restricted existential quantification is best modelled by taking

$$\exists x\big(P(x) \wedge A(x)\big)$$

to express 'some Ps are As'. But then, the de Morgan duality of restricted existential and universal quantification holds only when the conditional expressing restricted universal quantification is the material conditional. This leaves the study of restricted quantification with a number of interesting problems to be sorted out. Thankfully, in our case we need not worry with these problems: We can use the material conditional for restricting the quantifiers, because the relation $=_{\mathsf{bool}}$ behaves classically.

Definition 12.6 Let $\forall^c v_i A$ stand for $\forall v_i\big({\sim}\,\mathrm{Cl}(v_i) \vee A\big)$ and $\exists^c v_i A$ for $\exists v_i\big(\mathrm{Cl}(v_i) \wedge A\big)$.

We can prove the following result.

Lemma 12.9 *In any property-class model* $\mathfrak{N}$, $\mathfrak{N}(\forall^c v_n A)_x = \bigcap_{a \in C} \mathfrak{N}(A)_{x(n/a)}$, *and* $\mathfrak{N}(\exists^c v_n A)_x = \bigcup_{a \in C} \mathfrak{N}(A)_{x(n/a)}$, *where* C *is the subset of the domain* N *consisting of classes.*

Proof: First the universal quantifier. Translating out shorthand and applying evaluation clauses, we have $\mathfrak{N}(\forall^c v_n A)_x = \bigcap_{a \in N} \mathfrak{N}({\sim}\exists v_j(v_n =_{\mathsf{bool}} \mathsf{ext\_of}(v_j)) \vee A)_{x(n/a)}$. Now, we can split the join in two. Our evaluation of the formula is equal to

$$\bigcap_{a \in C} \mathfrak{N}({\sim}\exists v_j(v_n =_{\mathsf{bool}} \mathsf{ext\_of}(v_j)) \vee A)_{x(n/a)}$$
$$\cap \bigcap_{a \in N \setminus C} \mathfrak{N}({\sim}\exists v_j(v_n =_{\mathsf{bool}} \mathsf{ext\_of}(v_j)) \vee A)_{x(n/a)}.$$

The first conjunct is the value we are after. We need just ensure that the second conjunct is the top element of the propositional structure, and we have our result. But this is simple. $\bigcap_{a \in N \setminus C} \mathfrak{N}({\sim}\exists v_j(v_n =_{\mathsf{bool}} \mathsf{ext\_of}(v_j)) \vee A)_{x(n/a)} \geq \bigcap_{a \in N \setminus C} \mathfrak{N}\big(\forall v_j(v_n \neq_{\mathsf{bool}} \mathsf{ext\_of}(v_j))\big)_{x(n/a)}$, and by the behaviour of boolean identity, we have for any $a \in N \setminus C$, $\mathfrak{N}\big(\forall v_j(v_n \neq_{\mathsf{bool}} \mathsf{ext\_of}(v_j))\big)_{x(n/a)} = \bigcap_{b \in N} \mathfrak{N}\big(v_n \neq_{\mathsf{bool}} \mathsf{ext\_of}(v_j)\big)_{x(n/a)(j/b)} = \top$, since a, the denotation of v_n, is never a class.

The case for the existential quantifier is dual, and is left to the reader. ◁

This has the helpful corollary.

Corollary *In each theory $\mathbf{X}^{\langle : \rangle c}$ we can prove the de Morgan identities*

$$\vdash \forall^c x {\sim} A \leftrightarrow {\sim}\exists^c x A, \quad \vdash \exists^c x {\sim} A \leftrightarrow \forall^c x {\sim} A,$$

and the detachment principles

$$\vdash Cl(a) \wedge A(a) \to \exists^c x A(x), \quad Cl(a) \vdash \forall^c x A(x) \to A(a),$$

where x is free for a in A.

In this theory we can define the usual classes as follows: The empty class is $\varnothing = \{x : \perp\}$, the universal class is $V = \{x : \top\}$, the union of a and b is $a \cup b = \{x : x \in a \vee x \in b\}$ and their intersection is $a \cap b = \{x : x \in a \wedge x \in b\}$. The subclass relation $\subseteq$ can be defined in at least two ways. Firstly, we could set $a \subseteq b$ to be $\forall x(x \in a \to x \in b)$, allowing it to hold between properties in general. Instead, we may wish to restrict the relation to classes. In that case, define $a \subseteq b$ to be $Cl(a) \wedge Cl(b) \wedge \forall x(x \in a \to x \in b)$.)

Power classes are defined by setting $\mathcal{P}a = \{x : Cl(x) \wedge x \subseteq a\}$. Here we use the predicate $Cl(x)$ to restrict the range of elements of the class to classes. Clearly we have $\mathcal{P}V \subseteq V$, in a violation of Cantor's theorem (as one would hope to have on this conception of classes).

We can also define the class $\{x : Cl(x)\}$ of all classes. This is another self-membered class. Talking of self-membered classes, we can define $S = \{x : Cl(x) \wedge x \in x\}$, and $R = \{x : Cl(x) \wedge x \notin x\}$, the class of all self-membered and non-self-membered classes respectively. In each of our theories, we have $\vdash R \in R \leftrightarrow R \notin R$, but we cannot proceed from this to $\vdash R \in R$ or $\vdash R \notin R$. Because of this, we cannot show that $R \cup S = \{x : Cl(x)\}$. Even though R and S are the classes of self-membered and non-self-membered classes respctively, they do not jointly make up the class of all classes, because some classes are neither self-membered nor non-self-membered.

In this section we have shown how a theory of classes naturally arises within a theory of properties. Instead of continuing to develop the theory of classes by examining what purely class-theoretic results hold, we will how to embed a theory of sets into this theory.

12.4 A Second-Order Theory

To develop hierarchical set theories like **ZF** in our property/class theory, we need to extend the expressive power of our language. We will do this by adding monadic second-order quantification. This is quite simple in practice. We use the same relational structures as before, but we add monadic second-order quantification to the language (using variables V_0, V_1, ... , which we often abbreviate X, Y, Z, ...) and we use the obvious interpretation in the relational structure.

All we need do is extend the definition of the interpretation map $\mathfrak{A}(-)$ from a relational structure $\mathfrak{A}$ into its associated propositional structure. In Chapter 4, this map took a formula and a valuation (a list of domain elements which do the job of the denotation of any free first-order variables) in the expression. Now we need to extend this by adding another valuation which does the job of giving the denotation of any free second-order variables in the expression.

Definition 12.7 Given a relational structure $\mathfrak{A}$ with domain A and an associated propositional structure $\mathfrak{P}$ with domain P, a *second-order valuation* D is a list with elements from the set P^A of functions from A to P. A pair d, D is a *valuation pair* when d is a first-order valuation and D is a second-order valuation.

On our definition, a second-order variable will denote a class in the domain — where a class is represented by its characteristic function.

Now we can extend the definition of the map to one which takes a formula and a valuation pair to give a proposition. The definition is that of

Definition 12.8 Each relational structure $\mathfrak{A} = \langle A; \mathfrak{P}, R_\xi\ (\xi < \alpha_1), r_\xi\ (\xi < \alpha_2)\rangle$ determines a Tarskian *interpretation* of the language $\mathcal{L}$ in the following way: The interpretation is given by inductively defining two functions $\mathfrak{A}^t(-)_-$ and $\mathfrak{A}(-)_{-,-}$ The clauses are as follows:

$$\mathfrak{A}^t(v_n)_d = d_n, \quad \mathfrak{A}^t\big(f_\xi(t_1, \ldots, t_{\mu_2(\xi)})\big)_d = r_\xi\big(\mathfrak{A}^t(t_1)_d, \ldots, \mathfrak{A}^t(t_{\mu_2(\xi)})_d\big),$$

So $\mathfrak{A}^t(t)_d$ is the denotation of term t, given that the remaining free variables in t are given values by the list d, v_n taking the value d_n. The inductive clauses for formulae are more complex, but no more difficult.

$$\mathfrak{A}\big(F_\xi(t_1, \ldots, t_{\mu_1(\xi)})\big)_{d,D} = R_\xi\big(\mathfrak{A}^t(t_1)_d, \ldots, \mathfrak{A}^t(t_{\mu_1(\xi)})_d\big), \quad \mathfrak{A}\big(V_n t\big)_{d,D} = D_n(\mathfrak{A}^t(t)_d),$$

$$\mathfrak{A}(B \wedge C)_{d,D} = \mathfrak{A}(B)_{d,D} \cap \mathfrak{A}(C)_{d,D}, \quad \mathfrak{A}(B \vee C)_{d,D} = \mathfrak{A}(B)_{d,D} \cup \mathfrak{A}(C)_{d,D},$$

$$\mathfrak{A}(B \to C)_{d,D} = \mathfrak{A}(B)_{d,D} \Rightarrow \mathfrak{A}(C)_{d,D}, \quad \mathfrak{A}({\sim} B)_{d,D} = -\mathfrak{A}(B)_{d,D}, \quad \mathfrak{A}(\bot)_{d,D} = F,$$

$$\mathfrak{A}(\forall v_n B)_{d,D} = \bigcap_{a \in A} \mathfrak{A}(B)_{d(n/a),D}, \quad \mathfrak{A}(\exists v_n B)_{d,D} = \bigcup_{a \in A} \mathfrak{A}(B)_{d(n/a),D},$$

$$\mathfrak{A}(\forall V_n B)_{d,D} = \bigcap_{\alpha \in P^A} \mathfrak{A}(B)_{d,D(n/\alpha)}, \quad \mathfrak{A}(\exists V_n B)_{d,D} = \bigcup_{\alpha \in P^A} \mathfrak{A}(B)_{d,D(n/\alpha)}.$$

Thus $\mathfrak{A}(B)_{d,D}$ is the value of the formula B, given that the remaining free individual variables in B are given values by the list d, as before, and the remaining free predicate variables are given values in from list D, V_n taking value D_n.

Given the definition of an interpretation, we are free to define the notion of *satisfaction*: represented by $\vDash_{d,D}$, relativised to a valuation pair d, D. We take it that $\mathfrak{A} \vDash_{d,D} B$ if and only if $e \leq \mathfrak{A}(B)_{d,D}$. So, a sentence is satisfied by a model just when it comes out as logically true.

Clearly, if B is a sentence, the interpretation is independent of the valuation pair, and so is whether or not the sentence is satisfied in the relational structure. More generally, if the free variables of B are in $v_0, \ldots, v_n$, and $V_0, \ldots, V_n$ the interpretation of B depends only on the first $n+1$ elements of each valuation. In this case we will write $\mathfrak{A}(B)[d_0, \ldots, d_n; D_0, \ldots, D_n]$ for $\mathfrak{A}(B)_{d,D}$ and $\mathfrak{A} \vDash B[d_0, \ldots, d_n; D_0, \ldots, D_n]$ for $\mathfrak{A} \vDash_{d,D} B$.

This definition is a simple adaptation of classical monadic second-order logic to our non-classical setting. This addition gives us much greater expressive power, at the price of losing the recursive axiomatisability of the theory.

The Theory Defined Now we need not have a language-dependent comprehension scheme. Our naïve property theory can be axiomatised at a single stroke.

Definition 12.9 The second-order naïve comprehension scheme is this:

$$\forall X \forall x (Xx \leftrightarrow x \in \langle X \rangle)$$

where object $\langle X \rangle$ is the 'reification' of the denotation of the predicate X. The monadic second-order logic $\mathbf{X}^2$ with the addition of the naïve comprehension scheme is called $\mathbf{X}^{2\langle\rangle}$.

In order to interpret a theory like $\mathbf{X}^{2\langle\rangle}$ we must have a definition of the variable binding term operator $\langle - \rangle$. It is simple:

Definition 12.10 A *second-order variable binding term operator* $\langle - \rangle$ takes a second-order variable and denotes a domain element. It is interpreted in the relational structure by a map $\rho : P^A \to A$, which sets $\mathfrak{A}^t(\langle V_n \rangle)_{d,D} = \rho(D_n)$. (Note that this makes the denotations of terms depend on the values of second-order variables.)

So, we can construct our second-order structures.

Definition 12.11 A second-order property-class structure $\mathfrak{N} = \langle N; \mathfrak{P}, \varepsilon, \iota, \rho \rangle$ is a relational structure in which ε interprets $\in$, ι interprets identity, and ρ interprets the variable binding term operator $\langle - \rangle$. The structure must satisfy the second-order comprehension axiom $\forall X \forall x (Xx \leftrightarrow x \in \langle X \rangle)$.

The semantic structures we have set up have an interesting property. The denotation of $\langle X \rangle$ depends on the denotation of X alone. This means we can prove the following in our theories.

LEMMA 12.10 *In each theory* $\mathbf{X}^{2\langle\rangle}$ *we have* $\forall x(Xx \leftrightarrow Yx) \dashv\vdash \langle X \rangle = \langle Y \rangle$.

Proof: Let $\mathfrak{N}$ be a model of the second-order naïve comprehension scheme. Any evaluation D which allows $\mathfrak{N} \vDash_{d,D} \forall x(Xx \leftrightarrow Yx)$ must have the denotation of X identical to the denotation of Y, by the universality of the quantifier. This means that $\mathfrak{N}^t\big(\langle X \rangle\big)_{d,D} = \rho(\alpha)$ where $\alpha : N \to P$ is the denotation of X (and Y) under D. But $\rho(\alpha) = \mathfrak{N}^t\big(\langle Y \rangle\big)_{d,D}$ too, giving us our result.

Similarly, if $\mathfrak{N} \vDash_{d,D} \langle X \rangle = \langle Y \rangle$ then because $\mathfrak{N}(\nu_0 \in \langle X \rangle)_{d(0/n),D} = \mathfrak{N}(\nu_0 \in \langle Y \rangle)_{d(0/n),D}$ for any $n \in N$ (as $\langle X \rangle$ and $\langle Y \rangle$ have the same denotations by D), we must have $\mathfrak{N} \vDash_{d,D} \forall x(Xx \leftrightarrow Yx)$ by the extensionality axiom tying the value of Xx to that of $x \in \langle X \rangle$. ◁

This means that $\langle X \rangle$ depends only on the extension of X. If X and Y are co-extensional variables, their corresponding classes $\langle X \rangle$ and $\langle Y \rangle$ are identical. We can then mimic the ext_of function using $\langle - \rangle$ alone.

We can extend the language to add constructs of the form $\langle x : A(x) \rangle$ (as in the first-order theory) interpreted in the obvious way. All you need do is note that $A(x)$ determines a function from the domain to the propositional structure, just like a free second-order variable. The denotation of $\langle x : A(x) \rangle$ to be ρ of that function. Then we can define ext_of(y) to be $\langle x : x \in y \rangle$, and this satisfies the properties of ext_of by the above result.

The Embedding Now to embed a classical set theory into our models of $\mathbf{X}^{2\langle\rangle}$ we need to be able to talk about the objects in our models. For this, we make use of a standard construction. Fix a model $\mathfrak{N}$. For each object c in $\mathfrak{N}$, add a constant c^* denoting c to the language $\mathcal{L}$

The power of the second-order theory will be made obvious in the results below. We will see that *any* sensible model (where sensibility is to be defined below) of a well-founded set theory resides inside any model $\mathfrak{N}$ of our second-order theory. In this way our theories $\mathbf{X}^{2\langle\rangle}$ are able to recapture all of the results of classical set theories.

Definition 12.12 $\langle Z; \epsilon \rangle$ is a *sensible* model of a classical set theory if and only if it is a model of some set theory, and if there is a well-ordering $\leq$ such that for each $z \in Z$, if $z' \epsilon z$ then $z' \leq z$.

Our sensible models are well-founded ones — that is, there are no infinite descending ϵ chains — and we demand only that they be well-orderable, in a way that reflects the construction of sets. If we allow Zorn's Lemma, then it is easy to see that any well-founded model $\langle Z; \epsilon \rangle$ can be equipped with a well-ordering $\leq$ which respects the membership relation in the appropriate way. Pick an arbitrary well-ordering of Z, $\leq'$. For the $\leq$-first object, pick the empty set. Then, given that we have picked the elements $A \subseteq Z$ as ordered by $\leq$, as the $\leq$-*next* element pick the $\leq'$-first element not yet picked, and which has its members in A. Since $\langle Z, \epsilon \rangle$ is well-founded, there will be such an element, unless $A = Z$. In this case, the selection process is finished. If not, continue up the well-ordering $\leq'$.

It is helpful to treat $\langle Z; \epsilon \rangle$ as a relational structure on the boolean algebra $2 = \{T, F\}$, where the T and the F are the same objects as the top and bottom elements in our propositional structures. In this way we can compare truth values in structures simply, because $\{T, F\}$ in our arbitrary propositional structures do the job of the two values in the two element boolean algebra.

THEOREM 12.11 *Given a second-order model $\mathfrak{N}$ of naïve comprehension, and* any *sensible model $\langle Z; \epsilon \rangle$ of a classical set theory, then there is a constant $\mathbf{Z}$ in $\mathfrak{N}$ which represents the model Z, such that for any formula A, $\langle Z; \epsilon \rangle \models A$ if and only if $\mathfrak{N} \models A^{\mathbf{Z}}$ where $A^{\mathbf{Z}}$ is A with its quantifiers restricted to $\mathbf{Z}$.*

Proof: Let $\langle Z; \epsilon \rangle$ be a sensible well-founded model of some classical set theory, with one empty set. Let $\leq$ be its well-ordering. Let $\mathfrak{N}$ be our second-order model of naïve comprehension. We will gradually extend the language of $\mathfrak{N}$ with a constant z^* for each $z \in Z$, and finally for Z itself. This proceeds by induction.

Firstly, for the base case, let $z_\varnothing$ be the $\leq$-least element in Z. It is the empty set, because it has no members. Let $a_\varnothing$ be $\mathfrak{N}(\langle V_0 \rangle)[\alpha]$ where $\alpha : N \to P$ by setting $\alpha(a) = F$ always. That is, $a_\varnothing$ is the empty set in $\mathfrak{N}$. It is $\rho(\alpha)$ where α is the constant F map. Then $a^*_\varnothing$ is an $\mathfrak{N}$ constant denoting $a_\varnothing$.

We have picked an $\mathfrak{N}$ object to be represent the empty set in Z. Now we do the induction step, showing that we can pick representatives of other Z objects, given that we have representatives of all the earlier objects in the ordering.

For an induction on the well-ordering, consider the βth element of Z, z_β, for some ordinal β. Suppose that we have found representatives a_γ (with corresponding constants a^*_γ) for each z_γ where $\gamma < \beta$. Then for a^*_β, we reason like this. Let $\alpha : N \to P$ be given as follows:

$$\alpha(a) = \begin{cases} T & \text{if for some } \gamma < \beta,\ a = a_\gamma \text{ and } z_\gamma \epsilon z_\beta, \\ F & \text{otherwise.} \end{cases}$$

Then set $a_\beta = \rho(\alpha)$. By this construction, $a \, \varepsilon \, a_\beta = T$ if and only if $a = a_\gamma$ for some γ and $z_\gamma \epsilon a_\beta$, and $a \, \varepsilon \, a_\beta = F$ otherwise. So, for all ordinals in the well-ordering $\leq$, we can choose representatives a_γ of the z_γ in Z.

Finally, we want a constant representing the set Z itself. This is easy. Define $\mathbf{Z}$ by setting

$$\alpha(a) = \begin{cases} T & \text{if for some } \gamma,\ a = a_\gamma \\ F & \text{otherwise.} \end{cases}$$

Then set $a_\infty = \rho(\alpha)$, and let $\mathbf{Z}$ denote a_∞.

For what comes ahead, it is useful to let the $\mathfrak{N}$ image of Z be called A. That is, $A \subset N$ is the set $\{x : \exists\gamma(x = a_\gamma)\}$ of $\mathfrak{N}$ domain elements chosen to represent Z.

Now for the theory embedding result, we will show that for all first-order formulas B in the language of $\langle Z; \epsilon \rangle$ and for all valuations $d \in Z^\omega$, $\langle Z; \epsilon \rangle (B)_d = \mathfrak{N}(B^{\mathbf{Z}})_{f(d)}$, where $f : Z^\omega \to A^\omega$ is the natural isomorphism given by setting $f(\langle z_{\alpha_0}, z_{\alpha_1}, \ldots \rangle) = \langle a_{\alpha_0}, a_{\alpha_1}, \ldots \rangle$. The result is given by a simple induction on the complexity of formulae. For the base case, note that $\langle Z; \epsilon \rangle (\nu_i \in \nu_j)_d = z_{\alpha_i} \epsilon z_{\alpha_j} = a_{\alpha_i} \varepsilon a_{\alpha_j} = \mathfrak{N}(\nu_i \in \nu_j)_{f(d)}$, as desired.

The induction step for conjunction, disjunction, negation and implication is straightforward. The quantifiers provide the only difficulty. For the universal quantifier we have $\langle Z;\epsilon\rangle(\forall v_n B)_d = \bigcap_{z\in Z}\langle Z;\epsilon\rangle(B)_{d(n/z)} = \bigcap_{z\in Z}\mathfrak{N}(B)_{f(d(n/z))}$ by the induction hypothesis. However, for each $z \in Z$ there is exactly one $a \in A$ where $f(d(n/z)) = f(d)(n/a)$, so $\bigcap_{z\in Z}\mathfrak{N}(B)_{f(d(n/z))} = \bigcap_{a\in A}\mathfrak{N}(B)_{f(d)(n/a)}$. And since $\mathbf{Z}$ is a classical class marking out the members of A, we have $\bigcap_{a\in A}\mathfrak{N}(B)_{f(d)(n/a)} = \bigcap_{a\in N}\mathfrak{N}(v_n \notin \mathbf{Z} \vee B)_{f(d)(n/a)} = \mathfrak{N}\big(\forall v_n(v_n \notin \mathbf{Z} \vee B)\big)_{f(d)}$ as desired. The case for the existential quantifier is dual. ◁

This is an extremely powerful result. It shows that any set theoretical constructions that can be consistently performed can be performed within our theories $\mathbf{X}^{2\langle\rangle}$. This means that models of these theories must be *large*. However, the theory $\mathbf{X}^{2\langle\rangle}$ does not commit us to strong set theories unless they are consistent. The reason is as follows. The result we have proved shows that certian consistent set theories (those with sensible models) are proved to be consistent by our theories $\mathbf{X}^{2\langle\rangle}$. This does not mean that *inconsistent* set theories are proved consistent by $\mathbf{X}^{2\langle\rangle}$. Our result will only embed set theories with large cardinals within $\mathfrak{N}$ only if those set theories are consistent. If it somehow turns out to be inconsistent to assume the existence of certain large cardinals, this does not affect $\mathbf{X}^{2\langle\rangle}$ in any way, other than there being no models of those set theories within $\mathbf{X}^{2\langle\rangle}$ models. Similarly, if, perchance, **ZF** were to be inconsistent, then $\mathbf{X}^{2\langle\rangle}$ will not contain **ZF** models. However, the smart money being on the consistency of **ZF**, in all likelihood there are **ZF** models within $\mathbf{X}^{2\langle\rangle}$ models.

This result is not only powerful, but it coheres well with the considerations given in the first section, about the relationship between sets and classes. We argued that the of sets considered as constructed entities (given by some method of construction) are a subclass of the larger collection of classes as extensions of properties. This is borne out by our construction, in which sets turn out to be classes, with special properties. Similarly, we noted that the notion of 'set' as constructed object is vague. It depends on how far you take the construction to go and how many sets you take to be constructed at each stage of the process. This is reflected in our embedding result. There are a whole family of set theories embedded within each of our models. In every way, our theory $\mathbf{X}^{2\langle\rangle}$ accords with our pretheoretic desiderata. This is a good sign. However, not all is sunny: Clouds of paradox darken the horizon.

Excursus: Before getting on to paradox, there is one small point to be dealt with. The theories $\mathbf{X}^{2\langle\rangle}$ are each couched in a second-order logic. The fact that the logics are second-order gave us the expressive power to set up our theories by positing a single axiom. Of course, this also means that our theories are not axiomatisable, because the second-order logics themselves are not axiomatisable. Also, once we have the comprehension axiom, all the work done by second-order quantification can be done by way of first-order quantification over properties. So, this leads us to a possible alternative approach. Instead of using second-order quantification, we could restrict our attention to first-order logic, and study property/class structures which satisfy the *closure conditions* of the second-order structures. That is, we consider models such that for every function α from the domain N to the class of propositions P, there is a corresponding domain element in N, the property represented by the function α. The resulting theory would do everything that the second-order theory will do. Its only difference is the way it fails to be axiomatisable. In the second-order theory, the non-axiomatisability is restricted to the purely part of the theory encoding the principles of the second-order logic. The property/class-theoretic part of the theory is axiomatisable at a single stroke. This is not the case with the first-order theory. Other than that, there is little difference. It is only an historical accident that the theory is couched in second-order terms here. □

Paradox? Second-order property theoretic models like $\mathfrak{N}$ are exciting and powerful. They seem to be the best possibility for bringing together a decent account of properties, classes and sets. However, anyone who wishes to use power like this must wield it wisely. In our case, any proponent of a second-order naïve property theory must be thoroughly committed to the non-classical logic in which the theory is expressed. The moral of the theory is that the paradoxes must be handled in a logic without contraction. The only way a rich and powerful theory like $\mathbf{X}^{2\langle\rangle}$ can be coherently maintained is if the *metalogic* is also contraction-free. The reason for this is simple: The property- and class-theoretic paradoxes can be expressed in the metalogic of $\mathbf{X}^{2\langle\rangle}$ as easily as in the theory itself.

Consider a model $\mathfrak{N} = \langle N; \mathfrak{P}, \varepsilon, \iota, \rho\rangle$. If it is truly a model of $\mathbf{X}^{2\langle\rangle}$, then for each $\alpha \in P^N$ there is a *different* $\rho(\alpha) \in N$. We must have $\alpha(a) = a \,\varepsilon\, \rho(\alpha)$ for each $a \in N$, so different functions α require different representatives $\rho(\alpha)$ in the domain. This means that $|N| \geq |P^N| \geq |2^N|$. This contradicts Cantor's Theorem, according to which, a set ought be strictly smaller than its power set. We have a paradox in the metalanguage.

Or do we? We already know in our class theory that $V = \{x : \top\} \supseteq \mathcal{P}(V) = \{x : Cl(x) \wedge x \subseteq V\}$, where $x \subseteq y$. So, *proper classes* in our class theory already violate the result of Cantor's theorem. Clearly the universal class (which has everything as a member) contains each element of its power class. (This does not mean that *sets* in our theories need violate Cantor's Theorem. These are purely classical objects, and they satisfy classical results, like Cantor's Theorem.) The metatheoretic "Cantor's Paradox" tells us only that the models of our theories must have domains that behave like proper classes. This is not surprising really, because they are intended to *represent* genuine classes.

The fact that our metalogic must be contraction-free puts our proof of the embedding theorem in doubt. We have clearly used definitions relying on classical behaviour in our recursive definition of the class $\mathbf{Z}$. We define the constants a^*_γ recursively, using definition by cases. Definition by cases only works if the cases are truly exhaustive and exclusive. That is, if they behave classically. Recall the definition of a_β:

$$\alpha(a) = \begin{cases} T & \text{if for some } \gamma < \beta,\ a = a_\gamma \text{ and } z_\gamma \,\epsilon\, z_\beta, \\ F & \text{otherwise.} \end{cases}$$

Then set $a_\beta = \rho(\alpha)$. This definition only makes sense if for each a, either for some $\gamma < \beta$, $a = a_\gamma$, or not. That is, excluded middle must hold in this case, in the metalogic. Similarly, we cannot allow this proposition (that for some $\gamma < \beta$, $a = a_\gamma$) be both true and false, for then the definition by cases will also fail (we must define $\alpha(a)$ to be both

T and F). In other words, we require the proposition to be Boolean. How do we know this? The answer is simple. Provided that we admit boolean identity '$=_{\text{bool}}$' into the metalanguage, then we can assume in the metalanguage that for all x and y, either $x =_{\text{bool}} y$ or $x \neq_{\text{bool}} y$. Similarly, if $x =_{\text{bool}} y$ and $x \neq_{\text{bool}} y$, then absurdity follows. So, our recursive step, constructing a_β from the earlier a_γs is permissible because it relies only on identity and the behaviour of the classical model $\langle Z; \epsilon\rangle$, which we may assume behaves classically.

So, our proof does not seem to fail by our own lights. Despite the seeming strictures of non-classical reasoning, we have shown how to embed classical set theory and a decent class theory into a rich and powerful property theory. Clearly, there is much more to be done. Formalising the metalanguage to make absolutely clear which deductions are allowed is an obvious future task. We have used definition by recursion up a well-odering. This seems to be valid. Can it be proved to be so from simple principles, in our logics? Or must we add it to the theory?

Because the preferred theory is second-order, we will not be able to provide a complete recursive proof theory. Perhaps there is a perspicuous complete proof theory like that of arithmetic given by adding the ω-rule to Peano arithmetic. If such a proof theory could be found, it would be worthwhile to study it in some depth.

Another fruitful line of further research would be the relationships between our theory and other classical set and class theories. We have shown how to embed well-founded theories in our models. Is a similar construction possible for models of Aczel's set theories with anti-foundation? Is it possible to embed **NF** models in our theories? If we can do this, then our theories will provide a useful site to consider the relationships between classical set theories. The results we have so far show that moving to a non-classical logic in the light of the paradoxes need not mean abandoning classical insights. Classical set theories reside within a general class theory using a contraction-free logic.

Chapter 13
Getting Contraction Back 1

Small is beautiful.
— E. F. SCHUMACHER *Small is Beautiful* [139]

Now we will take a slightly different tack, to examine one way logics can fail to be contraction-free.

13.1 Comprehension and Implications

We have seen that for a logic (containing an implication operator $\to$, that satisfies *modus ponens*) to contain a naïve comprehension scheme, yet remain nontrivial, the rule of contraction must be absent. This condition is certainly necessary, but it is by no means sufficient. Consider again the proof of Curry's paradox, where $H_B = \{x : x \in x \to B\}$ for your favourite sentence B:

1	(1)	$H_B \in H_B$	A
1	(2)	$H_B \in H_B \to B$	$1 \in$E
1;1	(3)	B	1,2 MP
1	(4)	B	3 **W**
0	(5)	$H_B \in H_B \to B$	1,4 CP
0	(6)	$H_B \in H_B$	$5 \in$I
0	(7)	B	5,6 MP

A moment's thought reveals that there is nothing particularly special about the operator $\to$ used in the definition of H_B. To trivialise our theory all we need is *some* operator $>$ expressible in the language that has these three properties:

(1)	$p \to q \vdash p > q$
(2)	$p > (p > q) \vdash p > q$
(3)	$p, p > q \vdash q$

Table 13.1

Then, with $D_B = \{x : x \in x > B\}$ we have:

1	(1)	$D_B \in D_B$	A
1	(2)	$D_B \in D_B > B$	$1 \in$E
0	(3)	$D_B \in D_B \to (D_B \in D_B > B)$	1,2 CP
0	(4)	$D_B \in D_B > (D_B \in D_B > B)$	3 **(1)**
0	(5)	$D_B \in D_B > B$	4 **(2)**
0	(6)	$D_B \in D_B$	$5 \in$I
0	(7)	B	5,6 **(3)**

Given that we call a logic without the contraction rule *contraction-free*, the following definition lends itself for use:

Definition 13.1 An operator satisfying the three conditions given in Table 13.1 is said to be a *contracting implication*. If a logic contains *no* contracting implication it is said to be *robustly contraction-free*.

What we have shown is the following:

THEOREM 13.1 *A logic must be robustly contraction-free if it is to support a nontrivial naïve comprehension scheme.*

Moh Shaw-Kwei showed that none of the n-valued Łukasiewicz systems is robustly contraction-free [97], but to date, this result has not been extended to a larger class of propositional logics. In this chapter we show that *no* finitely valued logic satisfying certain general conditions is robustly contraction-free, and that some other logics also fail in this regard. Finally, we will show that each of our favourite logics is robustly contraction-free. To start, we will consider the logic **BN4**.

13.2 BN4

Recall **BN4** from Chapter 3, section 3. It is easy to show that contraction fails in **BN4**: $\mathsf{n} \to (\mathsf{n} \to \mathsf{b}) = \mathsf{t}$ but $\mathsf{n} \to \mathsf{b} = \mathsf{n}$.

But despite the fact that **BN4** is contraction-free, it fails to be robustly contraction-free. To see this, define a connective $>$ evaluated by the condition that $x > y$ is $x \wedge \mathsf{b} > y$ (where b appears in our language as t). It is easy to see that this connective satisfies conditions **(1)** and **(3)**. We are left with verifying **(2)**. But this is simple, given the table:

$>$	t	b	n	f
t	t	b	n	f
b	t	b	n	f
n	t	t	t	t
f	t	t	t	t

For $x > y$ to be undesignated, we must have x as either t or b, and y as either n or f. And in this case $x > y = y$. It follows that $x > (x > y) = x > y$, so $x > (x > y)$ is undesignated also. Contraposing this we see that $>$ contracts.

13.3 ... Finitely Valued Logics ...

As indicated before, the trouble with **BN4** is not restricted, but rather, it is suggestive of a problem that plagues all finitely valued logics. The problem can be explained like this. Given an implication functor $>$, we can define another implication functor in terms of it, but weaker — either as $x \wedge \tau > y$, for some true constant τ, or as $x > (x > y)$, or by some other means. However we do it, the new operator will satisfy *modus ponens*, and it will be weaker that $>$, in some sense to be explained later. We continue this weakening process *ad infinitum*, and eventually (finitely valued logics being the cramped places that they are) we shouldn't get anything new. Once this happens, we get contraction. This is the guiding idea in what follows.

We will consider a finitely valued logic, presented as a propositional structure on a finite set V (see Chapter 2 for the definition of a propositional structure). In fact, the conditions we need on our logic are considerably weaker than those given in Chapter 2. Define the structures we can consider, below:

Definition 13.2 A set V will define a *semilattice logic* if and only if it satisfies the following conditions:

- There is an operator $\wedge$ on V that defines a semilattice ordering $\leq$ on V. In other words, $\wedge$ is idempotent, symmetric, and associative, and $x \leq y$ iff $x \wedge y = x$.
- There is a set D of *designated* elements in V, which forms a filter. In other words, if $x \in D$ and $x \leq y$ then $y \in D$, and if $x, y \in D$ then $x \wedge y \in D$.
- The conjunction of all elements of D (itself an element of D) can be *named* in the language. We will call it t. Then t is the $\leq$-least element of D.
- There is an operator $\rightarrow$ that satisfies $x \leq y$ iff $t \leq x \rightarrow y$.

We will consider a given finite semilattice logic, and show that it can express a contracting implication. The first result is a lemma concerning the behaviour of $\rightarrow$.

LEMMA 13.2 *If* $t \leq x \rightarrow y$ *then* $t \leq x \wedge z \rightarrow y$.

Proof: If $t \leq x \rightarrow y$ then $x \leq y$ and so $x \wedge z \leq y$ by the properties of semilattices. This gives $t \leq x \wedge z \rightarrow y$. ◁

We define an infinite family of operators:

Definition 13.3 For each $n = 0, 1, 2, \ldots$, define $>_n$ on V by fixing

$$x >_0 y = y \qquad x >_{n+1} y = x \wedge t \rightarrow (x >_n y)$$

It is easy to show that for each m, n, $x >_m (x >_n y) = x >_{m+n} y$.

LEMMA 13.3 *If* $t \leq y$ *then* $t \leq x >_1 y$.

Proof: If $t \leq y$ then $t \leq t \rightarrow y$, and hence $t \leq x \wedge t \rightarrow y$ by lemma 13.2. ◁

LEMMA 13.4 *If* $t \leq x >_n y$ *then* $t \leq x >_{n+1} y$, *for each* $n = 0, 1, 2, \ldots$

Proof: This follows from lemma 13.3, as $x >_{n+1} y = x >_1 (x >_n y)$. ◁

LEMMA 13.5 *If* $t \leq x \rightarrow y$ *then* $t \leq x >_n y$ *for each* $n > 0$.

Proof: If $t \leq x \rightarrow y$ then $t \leq x \wedge t \rightarrow y = x >_1 y$. Then $n - 1$ applications of the previous lemma gives us the result. ◁

In other words, each $>_n$ satisfies condition **(3)**. Furthermore, we can show that $>_n$ satisfies condition **(1)**.

LEMMA 13.6 *If* $t \leq x >_n y$ *and* $t \leq x$ *we also have* $t \leq y$.

Proof: Clearly this result holds for $n = 0$. If it holds for n, note that $t \leq x >_{n+1} y = x \wedge t \rightarrow (x >_n y)$, so *modus ponens* for $\rightarrow$ (using $t \leq x \wedge t$) gives $t \leq x >_n y$, and our hypothesis gives us the result. ◁

Each of these results work in *any* semilattice logic. The finiteness condition is used in the following:

LEMMA 13.7 *For some* n, $t \leq x >_n y$ *if and only if* $t \leq x >_{n'} y$ *for each* $n' > n$.

Proof: For convenience, name the m elements of V as $x_1, \ldots, x_m$. For each implication $>_n$, consider the $m \times m$ matrix A_n, where the (i, j) element is 1 if and only if $x_i >_n x_j$ is designated, and is 0 otherwise. In other words, $[A_n]_{ij} = 1$ if and only if $t \leq x_i >_n x_j$, and $[A_n]_{ij} = 0$ otherwise. By lemma 13.4, the sequence $A_0, A_1, A_2, \ldots$ is a monotonic sequence of matrices, in that for each i, j, once the (i, j) element gets the value 1, it keeps that value in every matrix in the sequence. It follows that for some n, $A_{n'} = A_n$ for each $n' > n$. To see this, take n_{ij} to be the least n where $[A_n]_{ij} = 1$, or let it be 0 if there is no such n. Take $n = \max(n_{ij})$, this is the desired number.

But if $A_{n'} = A_n$ for each $n' > n$, it follows that $t \leq x >_n y$ if and only if $t \leq x >_{n'} y$ for each $n' > n$, as desired. ◁

Given this, we can prove our major theorem.

THEOREM 13.8 *Any finitely valued semilattice logic is not robustly contraction-free.*

Proof: Let n be such that $t \leq x >_n y$ if and only if $t \leq x >_{n'} y$ for each $n' > n$. Then $t \leq x >_n y$ if and only if $t \leq x >_{2n} y$. But this is simply $t \leq x >_n (x >_n y)$, and so, $>_n$ satisfies condition **(2)**. But by previous lemmas $>_n$ also satisfies conditions **(1)** and **(3)**. ◁

It follows that *any* finitely valued semilattice logic is useless for the project of formalising a nontrivial naïve comprehension scheme.

13.4 ... and Others

Other contraction-free logics also fail to be robustly contraction-free, despite the fact that they are not finitely valued. For example, Abelian Logic (called **A**, and introduced in Meyer and Slaney's 'Abelian Logic from A to Z' [96]) also fails to be robustly contraction-free. **A** can be defined by way of a particular propositional structure — on the set Z of integers. The lattice ordering is the obvious one on Z, and so, conjunction and disjunction are min and max respectively. The implication $x \to y$ is simply $y - x$, and so, the designated values are the non-negative integers. Any number of negations can be defined, but the canonical example is given by taking $\sim x$ to be $x \to 0$, which is simply $-x$. A simple check verifies that **A** properly contains **C**. In fact, it is given by adding to **C** the axiom:

$$((A \to B) \to B) \to A$$

which can be seen as a generalised double negation axiom. One interesting fact about **A** is that it has *no* nontrivial finite propositional structures as models. (To show this, note that the presence of an object F such that $F \leq x$ for each x in a structure will lead to triviality. This is because $(x \to F) \to F$ is equal to $F \to F$, which is T (the top element, $\sim F$) but $(x \to F) \to F \leq x$, so $T \leq x$.)

It is also a simple exercise to show that $>_n$ will not contract for each $n = 1, 2, \ldots$ So, to show that **A** is not robustly contraction-free, we must take another line. We must find a function $f : Z^2 \to Z$, satisfying:

(1a)	If $x \le y$ then $f(x, y) \ge 0$
(2a)	If $f(x, f(x, y)) \ge 0$ then $f(x, y) \ge 0$
(3a)	If $f(x, y) \ge 0$ and $x \ge 0$ then $y \ge 0$

Table 13.2

where f has been defined in terms of addition, subtraction, min, max and zero. The associated logical operator will then satisfy the conditions required for a contracting implication.

The first thing to note is that a function f satisfying $f(x, y) < 0$ iff $x > 0$ and $y < 0$ will satisfy these conditions. (This is a simple verification, and is left to the reader.) The challenge is to define such a function in terms that are allowed. One such function is given as follows:

$$f(x, y) = \max(y, 0) + \max(-x, 0) + \min(\max(-x, y), 0)$$

It is reasonably simple to check that it satisfies the conditions **(1a)** to **(3a)**. The corresponding implication operator is the horrible looking:

$$A > B = (B \vee 0) \circ (\sim A \vee 0) \circ ((\sim A \vee B) \wedge 0)$$

This 'simplifies' to

$$A > B = \sim((\sim A \vee B) \wedge 0 \to ((B \vee 0) \to (A \wedge 0)))$$

It follows that while **A** is contraction-free in that no simple minded nesting of arrows will yield a contracting implication, a more devious approach will give one. This failure of **A** to be robustly contraction-free is somewhat surprising, for it explicitly contains an infinite number of truth-values, which ought to be enough to distinguish any number of repetitions of premises. What strikes this observer is that some form of disjunctive syllogism might be lurking in **A**, allowing the operation $>$ to detach. (After all, $A > B$ does contain $\sim A \vee B$ as a subformula.) However, this is all speculation.

13.5 A Generalisation

Curry paradoxes are not just given by operators $>$ that satisfy contraction in the sense of condition **(2)**. If we have an operator $>$ that satisfies **(1)**, **(3)** and the rule from $A > (A > (A > B))$ to $A > (A > B)$, then a naïve comprehension scheme is also trivialised. This could be called 3–2 contraction, instead of the 2–1 contraction of condition **(2)**. In general, an operator satisfying **(1)**, **(3)** and $(N+1)$–N contraction is enough to trivialise a naïve comprehension scheme. It might be thought that this gives us a new way of trivialising the scheme, but this is not the case. We will show that this doesn't add anything new, as any 2–1 contraction-free logic is also $(N+1)$–N contraction-free. To do this, we expand our definition a little.

Definition 13.4 Let $A >^0 B$ be B and let $A >^{n+1} B$ be $A > (A >^n B)$. (Note that this definition differs from the previous kind of iterated implication, in that t is absent.) Clearly $A >^{N+M} B = A >^N (A >^M B)$. The implication $>$ is an M–N *contracting operator* (where $M > N$) if the rule from $A >^M B$ to $A >^N B$ is truth preserving.

This is enough for our preliminary result.

LEMMA 13.9 *If $>$ is an $(N+1)$–N contracting operator then it is also an M–N contracting operator for each $M > N$.*

Proof: Prove the result by induction on M. The base case, $M = N+1$ holds by definition. Suppose that we have $A >^M B \vdash A >^N B$, for $M > N$. Then $A >^{M+1} B = A >^{N+1} (A >^{M-N} B) \vdash A >^N (A >^{M-N} B) = A >^M B$, by the fact that $>$ is $(N+1)$–N contracting, and by the induction hypothesis, this gives $A >^N B$, as desired. ◁

LEMMA 13.10 *If $>$ is an $(N+1)$–N contracting operator in a logic, then $>^N$ is a 2–1 contracting operator in that logic.*

Proof: By the lemma, $>$ is $2N$–N contracting, so we have $A >^{2N} B \vdash A >^N B$, so $A >^N (A >^N B) \vdash A >^N B$ as desired. ◁

So, if we have an $(N+1)$–N contracting operator that also satisfies **(1)** and **(3)**, we have an associated operator that satisfies **(2)**, and also **(3)** (as can easily be checked). What is not so obvious is that it will satisfy **(1)**. If $>^{2N}$ does not satisfy **(1)** it is easy to define an operator that does. Take $\Rightarrow$ to be given by setting $A \Rightarrow B = A \wedge t >^{2N} B$. By Lemmas 13.2 to 13.6, this operator will satisfy **(1)** and **(3)**, and **(2)** follows from the validity of contraction for $>^{2N}$. So, we have the following result:

THEOREM 13.11 *If a semilattice logic is 2–1 contraction-free, it is also $(N+1)$–N contraction-free.*

13.6 The Trouble Avoided

Thankfully it is clear that many logics *are* robustly contraction-free. One such is Łukasiewicz's infinitely valued logic. In it there are no contracting implication operators, as can be seen by the fact that the naïve comprehension scheme is consistent in that logic. It follows that any of its sublogics — including *all* of our favourite logics — are robustly contraction-free.

We will end this short chapter with a conjecture. It does not look particularly easy to prove, but any headway made on it would be most welcome.

CONJECTURE *A logic is robustly contraction-free if and only if it nontrivially supports a naïve comprehension scheme.*

Obviously this will work only given that the logic satisfies a few small conditions — namely that it can express the comprehension scheme. We have shown that robust contraction-freedom is *necessary* for nontriviality. The hard part is proving sufficiency. This is where you, gentle reader, have your turn.

13.7 Note

An ancestor of this chapter has been accepted published by *Studia Logica* [126]. I wish to thank the anonymous referee for helpful hints for clarifying my prose, and for pointing me in the direction of the generalisation.

Chapter 14
Getting Contraction Back 2

A *Modal* Proposition
may be stated as a *pure* one,
by *attaching the mode*
to one of the Terms.
— WHATELY *Elements of Logic* [166]

14.1 The Problem of Communication

The reasoner who rejects contraction is a deviant. There is no doubt about this. Nearly all of the propositional logics given formal treatment in the last century have counted contraction as a valid rule. So, those that take this rule to be invalid are truly deviating from the norm. This raises a problem, for *some* of these standard logical systems have worthwhile insights. For example, we take it that a reasoner using the relevant logic **R** to ground inference is not being totally unintelligible — we just disagree about the general applicability of some of inference rules that are used. One way to lessen the negative impact of our deviance would be a way of representing the worthwhile insights of standard logical systems within our contraction-free systems. For example, this would provide a way to translate **R**-valid arguments into our own language, and show them to be valid according to our logic under this translation. In this way, some kind of bridge is formed between two parties, and contraction-free logicians will be able to use the insights gained from their "contraction-added" friends.

The task at hand is similar to Bob Meyer's work in his "Intuitionism, Entailment, Negation" [86]. There he shows that the theorems of intuitionistic logic can be represented inside the relevant logic **R**. The representation using the machinery of propositional quantification defines $A \supset B$ as

$$(\exists p)(p \wedge (A \wedge p \to B))$$

and in the presence of t we can define $A \supset B$ as simply

$$A \wedge t \to B$$

Given either representation '$\supset$' expresses intuitionistic implication. Furthermore, given a suitably "absurd" constant $\perp$, defining $\neg A$ as $A \supset \perp$ provides a counterpart to intuitionistic negation. Meyer's result is that $\langle \neg, \wedge, \vee, \supset \rangle$ fragment of **R** is intuitionistic logic.

There is something pleasing in this result — a proposition A intuitionistically "implies" B provided that there is a true proposition p such that the conjunction of A and p *relevantly* implies B. The justification for this is that p will contain the extra information required to *relevantly* deduce B from the conjunction of A and p.

A similar story can be told in the case of contraction-free logics, except that instead of using the machinery of propositional quantification, we add a modal operator to our language.

14.2 The New Modality

This modal operator will allow contraction in restricted contexts. Our account will draw from work on modalities in relevant logics, and on Girard's linear logic [51, 52]. To make our debt to the latter work obvious, we will denote our modality by the label '!.' There is some question as to how '!' should be vocalised. Possibilities are: 'bang!' (for the computer scientists), or 'shriek' (for the mathematicians), 'of course' (for the linear logicians) or simply, 'necessarily' (for the rest of us). The operator '!' will at least satisfy the basic conditions for an **S4**-like modality. These are

$!T$	$!A \to A$	$!\wedge$	$!A \wedge !B \to !(A \wedge B)$
$!4$	$!A \to !!A$	$! \to$	$!(A \to B) \to (!A \to !B)$
$!\exists$	$\exists x!A \to !\exists xA$	$!\forall$	$!\forall xA \to \forall x!Ax$
$!N$	$\dfrac{\vdash A}{\vdash !A}$		

Table 14.1

If you add these axioms and rule to a logic **X**, you get the standard (**S4**-like) logic **X4** (thus there are logics **DW4**, **TW4** and so on). Adding to these the axiom

$$!5 \quad A \to !?A$$

(Where '?' is shorthand for '$\sim ! \sim$' — it expresses the notion of possibility.) '!' acquires **S5** like properties. The resulting logics are called **DW5**, **TW5** and so on. None of these are particularly interesting in terms of contraction. The interest in this chapter results from the addition of the axiom

$$!WI \quad !A \to !A \circ !A$$

Now, a modalised claim ($!A$) contracts (it gives $!A \circ !A$). This does not give us *full* contraction — as we shall see — but it allows contraction of a limited class of formulae. This is interesting, because it enables us to model contraction-added systems within their contraction-free counterparts, while avoiding the collapse of full contraction.

Adding $!WI$ to the systems **DW4–CK4** gives the systems **DW!4–CK!4**. Similarly, adding them to the **S5** systems gives **DW!5–CK!5**.

LEMMA 14.1 *In* **DW4**, *and in all stronger systems,* $\vdash !(!A \vee !B) \leftrightarrow !A \vee !B$.

Proof: One arm of the biconditional is an instance of $!T$. To prove the other, note that $\vdash !A \to !A \vee !B$, and so by $!N$, $\vdash !(!A \to !A \vee !B)$, which by $! \to$ gives $\vdash !!A \to !(!A \vee !B)$, and so $!4$ gives $\vdash !A \to !(!A \vee !B)$. Similarly, we have $\vdash !B \to !(!A \vee !B)$, and $\vee$ *Elimination* gives us the result. ◁

LEMMA 14.2 *In* **DW4**, *and in all stronger systems,* $\vdash\ !(!A \circ !B) \leftrightarrow !A \circ !B$.

Proof: As before, one arm of the biconditional is an instance of $!T$. To prove the other, note that $\vdash\ !A \to (!B \to !A \circ !B)$, by the definition of fusion. Then $!N$ gives $\vdash\ !(!A \to (!B \to !A \circ !B))$, so $\vdash\ !!A \to !(!B \to !A \circ !B)$ by $!\!\to$. Using $!\!\to$ again gives $\vdash\ !!A \to (!!B \to !(!A \circ !B))$. The fact that $\vdash\ !C \leftrightarrow !!C$, and the substitutivity of equivalents then delivers the result. ◁

The facts just proved will be referred to as $!\vee$ and $!\circ$ respectively.

One thing that is important to note is that our modality differs from '!' in the context of linear logic. For '!' in linear logic satisfies:

$$!K \quad !A \circ !B \to !A$$

as well as the **S4** postulates and $!WI$. Adding this postulate to a logic like **C!4** is undoubtedly interesting, but it would also make the logic irrelevant. Finding an example is a simple exercise. Here's a hint. Unpack the axiom, rewriting the fusion as a nested conditional — we get $\vdash\ !A \to (!B \to !A)$. Pick $!A$ to be a theorem, and $!B$ to be irrelevant to $!A$. We get $\vdash\ !B \to !A$ by *modus ponens*. So, in a real sense linear logic with exponentials is *not* a relevant logic, despite its obvious family resemblance to the logics we are studying here.

Before continuing, we ought to say a few words about the interpretation of our modality. This clearly hinges on the reasons one has to believe that contraction fails in general. If we take the vagueness account of contraction, it seems that the reason contraction fails is that $A \circ A$ "stretches the truth" further than A alone. If this interpretation holds sway (as it does in the 'fuzzy logic' interpretation of the Łukasiewicz many valued logics), we could take $!A$ to be 'it is determinately the case that A,' or 'it is definite that A.' Other interpretations for ! will be appropriate in cases where contraction fails for other reasons.

In the case of vagueness, the other axioms for '!' are not implausible, although one might be unsure about $!4$ and $!5$. We will stick with them for the moment, leaving the consideration of their removal at a later stage.

We will now prove various results about the modality in these logics, and the way that they allow their 'contraction-added' counterparts to be represented within them. But before this, it is necessary to introduce some other notions from the study of relevant logics.

14.3 The Algebraic Analysis

To model the **S4** modality in a propositional structure (see Chapter 3), it is sufficient to add a unary predicate J, that satisfies the following conditions

JT	$Ja \le a$	$J\cap$	$J(a \cap b) = Ja \cap Jb$
$J4$	$Ja = JJa$	$J\cup$	$J(Ja \cup Jb) = Ja \cup Jb$
Je	$Je = e$	$J\cdot$	$J(Ja \cdot Jb) = Ja \cdot Jb$
$J\bigcap$	$J(\bigcap X) \le \bigcap J(X)$	$J\bigcup$	$\bigcup J(X) \le J\bigcup(X)$

Table 14.2

The soundness and completeness proof for these is quite straightforward — soundness is simply tedious, and completeness follows the usual Lindenbaum construction. (Showing that the postulates for J work is easy, given $!\vee$ and $!\circ$.) The details are left as an exercise. To model the **S5** structure, it is enough to add that:

$$J5 \quad a \leq J\overline{J\overline{a}}$$

And finally, to model $!\,WI$ we add that:

$$JWI \quad Ja \leq (Ja)^2$$

Instead of referring to J, it is possible to define the modality by specifying a set $\mathcal{O}$, of "open" elements in the lattice. These correspond to the elements a such that $Ja = a$. The class $\mathcal{O}$ is fixed to be a sublattice of the structure, containing e and F (a minimal lattice element) and closed under fusion. Then, we can define Ja for arbitrary a, to be $\bigvee\{x : x \leq a \text{ and } x \in \mathcal{O}\}$. (For this definition to work, we need that the structure be closed under suitable joins. This is no problem, as the soundness and completeness proofs work for *complete* propositional structures.) To model **S5** conditions, we need simply make $\mathcal{O}$ closed under negation, and to model $!\,WI$, we ensure that elements of $\mathcal{O}$ satisfy $a \leq a^2$.

Theorems and Proofs Given this model structure, it is simple to prove interesting theorems concerning the modal systems. Firstly, the **S4**, **S5** and **!4** systems are conservative extensions of their base systems.

THEOREM 14.3 *Given a propositional logic* **X**, *with a sound and complete modelling in terms of propositional structures,* **X4**, **X5** *and* **X!4** *are conservative extensions of* **X**.

Proof: Take a formula A that is not a theorem of **X**. By our condition on **X**, there is a propositional structure $\mathcal{P}$, and an interpretation h on $\mathcal{P}$, such that $h(A)$ is not in the positive cone. It suffices to define a set $\mathcal{O}$ of opens on $\mathcal{P}$, that satisfy the conditions for **X4**, **X5** and **X!4**. For the first two cases, take $\mathcal{O}$ to be the set of *all* the elements of $\mathcal{P}$. This trivially satisfies each of the conditions.

For **X!4**, this will not do, for the condition that if $a \in \mathcal{O}$, then $a \leq a^2$ will not necessarily be satisfied. Instead, take $\mathcal{O}$ to be $\{e, F\}$. This is closed under $\cap$, $\cup$ trivially (as $F \leq e$), and almost trivially, under fusion. Clearly $e^2 = e$, and $F^2 \leq F \cdot e = F$, so $F^2 = F$. Finally, $F \cdot e \leq F$, because e works that way. $\mathcal{O}$ satisfies the contraction condition, as we have seen that $e^2 = e$ and $F^2 = F$. ◁

Unfortunately, this result does not extend to all of the **X!5** systems. The most we can hope for is the following theorem.

THEOREM 14.4 **CK!5** *is a conservative extension of* **CK**.

Proof: Take A, a non-theorem of **CK**. There is a **CK**ps $\mathcal{P}$, and an interpretation h on $\mathcal{P}$, such that $h(A)$ is not in the positive cone. As before, it suffices to find a set $\mathcal{O}$ of opens on $\mathcal{P}$, that is closed under the rules for **CK!5**. $\{e, F\}$ satisfies this — for in **CK**, $\overline{e} = F$, and so, this set is closed under negation. That this set is closed under the other connectives follows from the proof of the previous theorem. ◁

The fact that this doesn't work for all of the **X!5** systems is a result of the following result, for which we need a definition.

LEMMA 14.5 *In any logic* **X!5**, *where* **X** *is at least as strong as* **DW**, $\vdash_{\mathbf{X!5}} f \to f \circ f$.

Proof: In the models for **X!5**, $\mathcal{O}$ is closed under negation — hence as $e \in \mathcal{O}$ we have that $\overline{e} \in \mathcal{O}$. As a result, $\overline{e} \leq \overline{e}^2$, which gives us that $\vdash_{\mathbf{X!5}} f \to f \circ f$, as desired. ◁

This means that **X!5** is not a conservative extension of **X**, given that **X** does not prove that $f \to f \circ f$. And this holds for any **X** no stronger than **C**. So, we have the following results:

THEOREM 14.6 **X!5** *is not a conservative extension of* **X**, *provided that* $\not\vdash_{\mathbf{X}} f \to f \circ f$. *So,* **DW!5**,**TW!5**, *and* **C!5** *are not conservative extensions of* **DW**, **TW** *and* **C**.

It would be satisfying to show that this result extends to the t-free systems **X!5** by finding a t, !-free formula that gives is a theorem of **X!5** but not of **X**. Unfortunately, no such formula has been found at the time of writing.

As negative as the last result is, the conservative extension results that we do have are enough to show that contraction is not provable in *any* of the modal systems we are covering. So, their addition does no harm to their contraction-free structure. But, as we will see, it gives them the ability to recover some contraction, in a controlled manner.

14.4 Recovering Contraction

The modality we have introduced allows us to recover the contraction-added counterparts of our contraction-free logics. In the first subsection below we will deal with the positive translation — ignoring negation — and the next subsection adds negation.

The Positive Part Consider the translation $^\circ : \mathbf{Fml}(\mathbf{X{+}WI}^+) \to \mathbf{Fml}(\mathbf{X!4}^+)$ (Where $\mathbf{Fml}(\mathbf{X})$ is the collection of formulae in the logic **X**, and $\mathbf{X}^+$ is the positive (that is, negation free) part of the logic **X** — which is axiomatised by the negation free axioms of **X**, given in the first chapter.

- $p^\circ = p$ for propositional parameters p.
- $(A \wedge B)^\circ = A^\circ \wedge B^\circ$.
- $(A \vee B)^\circ = !A^\circ \vee !B^\circ$.
- $(A \to B)^\circ = !(A^\circ) \to B^\circ$
- $(A \circ B)^\circ = !A^\circ \circ !B^\circ$.
- $(\forall x A)^\circ = \forall x A^\circ$.
- $(\exists x A)^\circ = \exists x !A^\circ$.

This provides us a translation between the logical systems. It is simple to show the following result.

THEOREM 14.7 *If* $\vdash_{\mathbf{X+WI}^+} A$, *then* $\vdash_{\mathbf{X!4}^+} A^\circ$, *for* **L** *being one of* **DW**, **TW**, **C** *or* **CK**.

Proof: It is sufficient to show that each Hilbert axiom of $\mathbf{X}+\mathbf{WI}^+$ is a theorem of $\mathbf{X!4}^+$ when translated, and that rules preserve theoremhood. The rules are simple: $\wedge$ *Introduction*$_2$ is unchanged from system to system, and *Modus Ponens* is a simple consequence of $!T$ and *Modus Ponens* in $\mathbf{X!4}^+$.

For example, $(A \to A \circ A)^\circ$ is a theorem of $\mathbf{X!4}^+$, as it is $!A^\circ \to !(!A^\circ \circ !A^\circ)$, which is easily shown to be a theorem by way of $!WI$, $!4$ and $!T$. This deals with *Contraction*.

The other axioms are no more difficult. We will deal with $\circ$ *Definition*, $\vee$ *Elimination*, $\exists$ *Distribution* and $\forall \to$ *Distribution*. The rest are left as an exercise. Firstly, we will show that in $\mathbf{X!4}^+$, $((A \circ B) \to C)^\circ \dashv\vdash (A \to (B \to C))^\circ$. In other words, we need $!(!A^\circ \circ !B^\circ) \to C^\circ \dashv\vdash !A^\circ \to (!B^\circ \to C^\circ)$ — but this is a simple consequence of $!\circ$.

As for $\vee$ *Elimination*, we simply note that we wish to show that $\vdash ((A \to C) \wedge (B \to C) \to (A \vee B \to C))^\circ$ is a theorem, and this amounts to $!((!A^\circ \to C^\circ) \wedge (!B^\circ \to C^\circ)) \to (!(!A^\circ \vee !B^\circ) \to C^\circ)$. By $\vee$ *Elimination* in $\mathbf{X!4}^+$ we have $\vdash (!A^\circ \to B^\circ) \wedge (!B^\circ \to C^\circ) \to (!A^\circ \vee !B^\circ \to C^\circ)$, so $!T$ and the transitivity of the conditional, together with some prefixing will give us the result.

For $\exists$ *Distribution* note that $\vdash \forall x(!A \to B) \to (\exists x!A \to B)$ (for x not free in B) gives $\vdash \big(\forall x(A \to B) \to (\exists x!A \to B)\big)^\circ$ as desired. For $\forall \to$ *Distribution* it is simple to get $\vdash \big(\forall x(A \to B) \to (A \to \forall x B)\big)^\circ$ because $\vdash \forall x(!A^\circ \to B^\circ) \to (!A^\circ \to \forall x B^\circ)$. This is enough working for us to confidently pronounce the result for $\mathbf{DW!}^+$.

As for the other logics, we need only prove that their additional axioms survive translation. We show the result for *Permutation*, and leave the rest as an exercise. We need to show that $\vdash \big(A \to (B \to C) \;\to\; B \to (A \to C)\big)^\circ$ in $\mathbf{X!4}^+$. That is, we need $\vdash !\big(!A^\circ \to (!B^\circ \to C^\circ)\big) \to \big(!B^\circ \to (!A^\circ \to C^\circ)\big)$. But permutation in the target logic gives $\vdash \big(!A^\circ \to (!B^\circ \to C^\circ)\big) \to \big(!B^\circ \to (!A^\circ \to C^\circ)\big)$ which with !T gives the result. ◁

The next thing to do is to show that the following result:

THEOREM 14.8 *If* $\vdash_{\mathbf{X!4}^+} A^\circ$, *then* $\vdash_{\mathbf{X}+\mathbf{WI}^+} A$, *for* $\mathbf{X}$ *being one of* **DW**, **TW**, **C** *or* **CK**.

Proof: Take the Hilbert-style proof of A° in $\mathbf{X!4}^+$, and remove all occurrences of '!' in that proof. The result is a proof of A in $\mathbf{X}+\mathbf{WI}^+$. That this is so is simple — each occurrence of a modal axiom becomes either an instance of *Identity*, or of **WI**. The other axioms remain unchanged, as do the rules. ◁

So, we have an injection $^\circ\colon \mathbf{X}+\mathbf{WI}^+ \to \mathbf{X!4}^+$, which maps the positive part of $\mathbf{X}+\mathbf{WI}$ into the positive part of **X!4** — in fact, it does more than this, as the observant may have noted. We have given transformation rules that allow one to transform a proof of A in $\mathbf{X}+\mathbf{WI}^+$ to a proof of A° in $\mathbf{X!4}^+$, and vice-versa. So, our proof is totally constructive. The next thing to do is to is to extend this result to negation.

Negation Negation adds a problem. Firstly, we need to decide what $({\sim} A)^\circ$ is going to be. The obvious candidate for ${\sim} A^\circ$ will not work, for we need results like *Contraposition*, and in translation, that would be $!(!A \to {\sim} B) \to (!{\sim} B \to A)$, which is simply not true, even in **S5**. (A Kripke countermodel can be given by two worlds, with B true in both, and A true in one, and false in another. In the latter world it is simple to check that $!B \to {\sim}A$ fails, but that $!(!A \to {\sim}B)$ is true.)

So, to make contraposition succeed, we need to make a different translation. One that succeeds is $({\sim}A)^\circ = {\sim}!A^\circ$. If we then expand our $^\circ$ in that way, to a mapping $^\circ : \mathbf{Fml}(\mathbf{X!5}) \to \mathbf{Fml}(\mathbf{X}+\mathbf{WI})$, we get the following results.

THEOREM 14.9 *If* $\vdash_{\mathbf{X+WI}} A$, *then* $\vdash_{\mathbf{X!5}} A^\circ$, *for* $\mathbf{X}$ *being one of* $\mathbf{DW}$, $\mathbf{TW}$, $\mathbf{C}$ *or* $\mathbf{CK}$.

Proof: The proof is just as for the positive results, except that we must also show that *Double Negation* and *Contraposition* work in their translated forms. We will work *Double Negation*, and leave *Contraposition* as an exercise.

The axiom translated is ${\sim}!{\sim}!A \to A$. Contraposing gives ${\sim}A \to !{\sim}!A$, and a double negation reveals it to be ${\sim}A \to !?{\sim}A$ — an instance of *!5*. ◁

We also get the next result.

THEOREM 14.10 *If* $\vdash_{\mathbf{X!5}} A^\circ$, *then* $\vdash_{\mathbf{X+WI}} A$, *for* $\mathbf{X}$ *being one of* $\mathbf{DW}$, $\mathbf{TW}$, $\mathbf{C}$ *or* $\mathbf{CK}$.

Proof: As before, erase all occurrences of ! from the proof of A°. It is enough to note that this is a proof in **X+WI** as before. The only thing to note is that the instances of *!5* become of the form $A \to {\sim}{\sim}A$, which is not an axiom of **X+WI**, but is readily proved. So, we expand the proof at these points with small subproofs of the double negation. ◁

The analogous result for **X!4** would be to show that the *intuitionistic* contraction-added counterpart **JX+WI** results from this translation. In fact, it is easily shown that at least **JX+WI** is embedded in **X!4**, but showing the converse is not easy. This is currently an open problem. (It seems that use of the Gentzen systems for these logics would facilitate a proof of this result, analogous to Schellinx's proof of the result for linear logic [138]. Unfortunately, the work in Gentzen systems for the *intuitionistic* counterparts of the logics we are interested in is presently near to nonexistent.)

14.5 Consequence: With Contraction and Without

In this section we give an explanation of one of the available notions of consequence in contraction-free logics. Then we show how our transformation of contraction-added systems into our systems transforms these notions.

In a contraction-free logic (as in many other logics) there are different notions of consequence. The only one we consider is that given by the natural deduction system. The sentence A follows from the bunch X of sentences just in the case where:

$$X \Vdash A$$

is provable. In other words, A follows from X just when $\vdash \mathcal{I}(X) \to A$. We will denote this relation by '$X \Vdash A$.'

Using our results it is simple to show that if in **X+WI** we have $X \Vdash A$, then in **X!5** we have $!X^\circ \Vdash A$, where $!X$ and X° are defined inductively in the obvious way:

• $!0 = 0$	• $0^\circ = 0$
• $!(X, Y) = !X, !Y$	• $(X, Y)^\circ = X^\circ, Y^\circ$
• $!(X; Y) = !X; !Y$	• $(X; Y)^\circ = !X^\circ; !Y^\circ$

(From this it is simple to show that $\vdash \mathcal{I}(!X^\circ) \leftrightarrow !\mathcal{I}(X)^\circ$, so $X \Vdash A$ in **X+WI** iff $\vdash \mathcal{I}(X) \to A$ in **X+WI**, iff $\vdash !\mathcal{I}(X)^\circ \to A^\circ$ in **X!5**, iff $\vdash \mathcal{I}(!X^\circ) \to A^\circ$ in **X!5** iff $!X^\circ \Vdash A^\circ$ in **X!5**.)

So, if our friend deduces A from X using a contraction-added system, we can join her by deducing A° (provided we agree on all other propositional axioms) in our system, using an **S5**-type modality. The only difference is that we have used a *different* bunch of premises (namely, $!X^\circ$). We have had to assume the *modal* versions of the premises accepted by our friend in non-modal guise. In practical terms, this will amount to our needing that our premises used be non-vague (or in some other way, able to contract). If this is the case, we may agree with our contraction-added friend — if not, we at least have the resources to see exactly where it is we disagree. But now the argument has shifted from a purely logical one of our refraining from some moves that our friend accepts, to one where we disagree about the truth of the *premises* of the argument at hand.

14.6 Triviality of Naïve Property Theory

Unfortunately, we will end this story on a somewhat sadder note. It is to be recalled that one of the interests of contraction-free logic is its ability to model naïve set theory, and other interesting things that are proved trivial by way of contraction. Our addition of contraction, even in its limited and controlled form, is enough to give a simple triviality result of naïve property theory founded on **DW!4** (and hence in any stronger logic).

THEOREM 14.11 **DW!4**$^{+\langle:\rangle}$ *is trivial.*

Proof: Let $C_B = \langle x : !x \in x \to B\rangle$ for your favourite proposition A. Then

$$\vdash (C_B \in C_B) \leftrightarrow (!(C_B \in C_B) \to B)$$

so by !T we have

$$\vdash !(C_B \in C_B) \leftrightarrow (!(C_B \in C_B) \to B)$$

and hence we have $\vdash !(C_B \in C_B) \circ !(C_B \in C_B) \to B$ and so by !WI we get

$$\vdash !(C_B \in C_B) \to B$$

which then gives $\vdash C_B \in C_B$ by the original biconditional. Then !K gives

$$\vdash !(C_B \in C_B)$$

which then gives $\vdash !B$. Unfortunately, B is quite arbitrary (nothing hangs on it being your favourite proposition) so, our theory is trivial. ◁

So, we have a sad result. It is clear that *each* of the modal rules apart from !4 or !5 are needed to get triviality. If we want to maintain !WI, we must reject one of the other rules. What is not clear is how one might restrict !T or !K to eliminate triviality, but retain the embedding results. Furthermore, the proof is available in the *intensional* (that is, $\wedge$, $\vee$ free) fragment of **DW!4**, which is a subsystem of the multiplicative and exponential fragment of linear logic, even upon the removal of the structural rules of commutativity and associativity. Exactly where this leaves contraction and its recovery is unclear at present.

Now seems not the time for further comment on this result, for more thought is needed concerning the significance of the formal results that we have seen. Furthermore, we have the open problems of embedding *intuitionistic* versions of the logical systems into the contraction-free **S4**-systems, and the need to give our modality a worlds semantics in the Routley-Meyer style. Each of these areas need further reflection, and some more theorems, but it now seems fitting to digest the results we have seen so far.

Chapter 15
Thoughts on Negation

. . . the capitalist mode of appropriation,
the result of the capitalist mode of production,
produces capitalist private property.
This is the first negation of individual private property,
as founded on the labour of the proprietor.
But capitalist production begets,
with the inexorability of a law of nature,
its own negation.
It is the negation of the negation.
— KARL MARX *Capital* [81]

The story has been long and involved. We have seen that logics lacking contraction are useful in a wide range of settings. We have seen a little of how theories in contraction-free logics perform, and we've been able to see more of what rejecting contraction gives us — and what it makes us do without. However, the story is not complete. The observant reader will have noticed that the treatment of negation in this work is quite ambivalent.[1] Insofar as negation is important to a treatment of the paradoxes, and insofar as the paradoxes are important to a treatment of logics without contraction, we ought to say a little more about negation and how it ought to feature in a decent logic.

15.1 Setting the Scene

It is somewhat surprising that negation has not featured very much in this work. After all, a great deal of the literature on the paradoxes and non-classical logic has focused on the way negation ought to be modelled. However, we have seen that the treatment of implication is also important, and that the structures arising out of a decent treatment of implication in the light of the paradoxes are also useful in other contexts. However, we need to say a little more about negation in order to fill out the account of the paradoxes. In this section we will set the scene, by sketching out the rival approaches to negation. In the next, we will consider some reasons to opt for one or the other of these approaches.

If we respond to the paradoxes by adopting a non-classical account of valid inference, it is clear that negation is one of the connectives which must operate non-classically. Given the machinery of naïve property theory, for example, it is simple to derive an equivalence

$$p \dashv\vdash {\sim} p$$

for a particular proposition p. Without using anything other than negation and the moves of property abstraction. Now if $\sim$ is a boolean negation, the theory is trivial. The proof to triviality usually takes two steps. Firstly, if $p \dashv\vdash {\sim} p$ then we have $p \vdash {\sim} p$ and hence $\vdash {\sim} p$ by the properties of Boolean negation. This in turn gives us $\vdash p$ by the original equivalence. These together give $\vdash q$ for an arbitrary q, and hence, triviality. There are two places this deduction could fail. Firstly, with the negations in our favourite logics, the deduction would not reach $\vdash {\sim} p$. We deny the step from $p \vdash {\sim} p$ to $\vdash {\sim} p$.

This treatment of negation is said to be *paracomplete*. Secondly, we could allow the deduction to $\vdash \sim p$ and $\vdash p$, without allowing the deduction to triviality. This treatment of negation is said to be *paraconsistent*.

Note that these deductions use negation, and no other connectives. A paracomplete account will deny the law of the excluded middle, to be sure (given $\vdash p \vee \sim p$ it is easy to get $\vdash \sim p$ from $p \vdash \sim p$) but this is because of the behaviour of negation, and not of disjunction. Similarly, a paraconsistent treatment of negation will not allow *ex falso quodlibet* ($p \wedge \sim p \vdash q$) but this is because of its treatment of negation, and not because of any strange properties of conjunction.

As I said before, the treatment of the negation paradoxes in our favourite logics is uniformly paracomplete. In none of our logics is the inference from $p \dashv\vdash \sim p$ to $\vdash \sim p$ allowed, given their treatment of negation. The question for us is, is that the correct approach to negation? This is especially important, because we have seen that the treatment of negation in our favourite logics is not particularly semantically natural, when it comes to the four-valued semantics. It will help to have some idea of the properties it is desirable for negation to have, in order to model it well. This is the task for this chapter.

Before we weigh the pros and cons of paraconsistency and paracompleteness, we must consider a possible third alternative. Terence Parsons has an interesting paper that expresses agnosticism between the two paraconsistency and paracompleteness [106]. He argues that because their treatment of the liar and like paradoxes is completely 'dual', there is nothing to choose between them. Partly this shows a lack of depth in his examples of paradoxes — he ignores the possibility that there might be paradoxes other than the liar and its close relatives, that the paraconsistent approach treats better than does the paracomplete approach, or *vice versa*. Also, there are other relevant criteria that may swing us towards one of paraconsistency and paracompleteness. Furthermore, if Parson's position is not simply agnosticism between the two positions, but another account in its own right, then he should adopt a formal system that is both paracomplete *and* paraconsistent. If that's the case, then it's hard to see what would move a reasoner to hold that the liar is both true and false, given that the naïve argument to that conclusion is invalid in the system that is paraconsistent and paracomplete, as the deduction from $p \vdash \sim p$ to $\sim p$ occurs *before* the contradiction arises. To get to the conclusion $p \wedge \sim p$, we need the further assumption $p \vee \sim p$ or something like it. If the reasoner does not have this at her disposal, it is not clear how this approach differs to the paracomplete approach, except for employing a logic that also happens to be paraconsistent. So, at first glance, Parson's proposal seems to actually favour the paracomplete solution, unless there is some other reason to believe that the liar is either true or false.

15.2 The Evidence

Taking Parson's 'agnosticism' to be dealt with, we can now take time marshalling the evidence for paraconsistency or paracompleteness.

Consistency. The paraconsistent approach to paradoxes like the liar commits us to the truth of contradictions. The paracomplete approach does not. I take this to be a point in favour of the paracomplete approach, for somehow, a violation of the law of non-contradiction seems just that bit more hard to swallow than a violation of the law of the excluded middle. Even committed advocates of paraconsistency hold that if we can get away with consistency at some point, then it is rational to do so. I agree. If a consistent approach to the paradoxes can be made to work, this is a reason to accept it, all other things being equal.

Uniformity. Graham Priest's recent work on the paraconsistent approach to the paradoxes has demonstrated an elegant uniformity to the approach [116]. In a nutshell, Priest shows that each of the *negation* paradoxes of self reference has the same form, and that the paraconsistent approach analyses each of them in the same way. (Namely, by accepting the argument to the self-contradictory conclusion and accepting it, but denying that from this conclusion, everything follows.)

It has not been shown that this is the case with the paracomplete solution. Some negation paradoxes (such as the paradox of well-foundedness, and the Burali-Forti paradox in set theory) do not progress by way of a proposition p such that $p \dashv\vdash {\sim} p$. So, a thoroughly paracomplete account of the paradoxes may not be able to stop the deduction in the same way as other negation paradoxes. It has to be shown that the argument fails at some other step, and that this is related in some way to the general contraction-free paracomplete programme. For example, there may be an instance of **PMP** or **W** that can be faulted.

EXCURSUS: Here is one example of how this might go. Consider Berry's paradox, which arises from simple arithmetic together with some facts about denotation. Consider this putative definition of a natural number:

> the least number not denoted by an English noun phrase with less than 100 letters

There are a finite number of English noun phrases with less than 100 letters, and assuming that every English noun phrase denotes either one natural number or no natural number at all (and this can be ensured by stipulation) only a finite number of natural numbers are so denoted. So, there is a least number not denoted by an English noun phrase with less than 100 letters. This is our number. However, it is, clearly, also denoted by an English noun phrase with less than 100 letters, by the reasoning we have just followed. This argument does not explicitly appeal to a contraction or excluded middle.

There are a number of facts used in this argument; facts about denotation, the size of the language, and so on. To give a detailed analysis of the argument, we must formalise it. Thankfully, this has already been done by Priest [113.5]. The entire formulation need not worry us now. Only the axioms for identity and the denotation relation D. Priest has the two axioms

$$\big(A(t_1) \wedge t_1 = t_2\big) \to A(t_2) \qquad (1)$$
$$D(\ulcorner t \urcorner, x) \leftrightarrow t = x \quad \text{for } t \text{ a closed term} \qquad (2)$$

where $\ulcorner t \urcorner$ is the numeral of the Gödel number of the term t. From these axioms, and others involving the behaviour of the least number operator and so on, Priest can derive the contradiction. We have already seen how our logics are quite sensitive to different ways to encode identity. It is an open question whether there is any notion of identity for which these two axioms are actually valid. My suspicion goes as follows: the paradox seems to show us that denotation is like truth, in the realms of arithmetic. We have no assurance that denotation behaves classically. So, we have no assurance that axiom (2) holds when identity is boolean. For boolean identity behaves classically. However, we have no assurance that axiom (1) holds when identity is not boolean. At most, our identity predicates are guaranteed to give us $t_1 = t_2 \vdash A(t_1) \to A(t_2)$. Axiom (1) looks too much like **PMP** for us to comfortably assume it in its generality. Once we admit that our identity can behave non-classically (which we must, if we tie it to denotation by axiom (2)) then we have to go further to motivate an axiom like (1), which seems to say that the antecedent $t_1 = t_2$ can be used to derive $A(t_2)$ from $A(t_1)$ without recording this use in an antecedent of its own. For those with contraction-free sensibilities, this is not to be done without at least a little further argument.

This discussion is merely the *start* of a real analysis of Berry's paradox from a contraction-free perspective. We have seen some idea of where *one* formalisation of the argument goes wrong on contraction-free lights. This does not mean that there are no other formalisations which do better. To respond to that, we need to add the theory of denotation to our arithmetics, see where Berry's argument fails, and interpret the result, giving us some idea of where the natural language argument also fails. This is an interesting research project, but it will not detain us now. □

The project of analysing all of the semantic paradoxes from a contraction-free perspective has not yet been completed. If it cannot be completed, this is a point for the paraconsistent solution, as a uniform solution to the paradoxes is more worthwhile than a number of unrelated ones.

More Uniformity. The consideration of uniformity of solution tilts the balance once we consider the paradoxes that don't involve negation. There are many paradoxes of this kind. What gives the similarity bite is the fact that granting *modus ponens*, the deviant approaches to the paradoxes have only one plausible place to fault the deduction: the step from $p \to\ p \to q$ to $p \to q$, the inference of *contraction*. The corresponding step in the negation case is the inference from $p \to {\sim} p$ to ${\sim} p$, which is where the *paracomplete* solution blocks the paradoxes. This similarity can be used in two ways.

If ${\sim} p$ is equivalent to $p \to f$ for some particular proposition f, then the two rules below are equivalent.

$$\frac{p \to {\sim} p}{{\sim} p} \qquad \frac{p \to\ p \to f}{p \to f}$$

The first is *reductio*, which the paraconsistent approach blesses as valid. (Intuitively, because of the truth of the law of the excluded middle.) The second is an instance of contraction, which we have seen must be rejected (in general) by both approaches. So, if we have this equivalence, it seems that the paraconsistent approach loses some of its uniformity — contraction is condemned as invalid in its generality, but a particular instance of it is permitted, and the problem with the liar paradox is located elsewhere.

An analysis that gives a uniform result for the liar and the Curried form of the liar is clearly superior than one which points the blame at different inferential moves.

However, we need to give some reason why $\sim p$ would be equivalent to $p \to f$ before convincing our paraconsistent friends. Holding that $\sim p$ is equivalent to $p \to f$ in almost every formal system known is not going to budge a paraconsistent logician, after all almost every formal system known is not paraconsistent, and our friend will have (weak) formal systems that invalidate the connection. As an example, Priest's logic Δ and Routley/Sylvan's favourite depth-relevant systems reject this principle [113,131] as do our logics at **EW** or below. Clearly if we have Def_t in the strong form

$$\vdash (t \to p) \leftrightarrow p \qquad (\mathrm{Def}_t)$$

we have our conclusion, by this short argument.

(1)	$\sim q \to (t \to \sim q)$	An instance of Def_t.
(2)	$(t \to \sim q) \to (\sim\sim q \to \sim t)$	Contraposition.
(3)	$(\sim\sim q \to \sim t) \to (q \to \sim t)$	As $q \to \sim\sim q$.
(4)	$\sim q \to (q \to \sim t)$	From (1), (2) and (3).
(5)	$(q \to \sim t) \to (\sim\sim t \to \sim q)$	Contraposition.
(6)	$(\sim\sim t \to \sim q) \to (t \to \sim q)$	As $t \to \sim\sim t$.
(7)	$(t \to \sim q) \to \sim q$	An instance of Def_t.
(8)	$(q \to \sim t) \to \sim q$	From (5), (6) and (7).
(9)	$\sim q \leftrightarrow (q \to \sim t)$	From (4) and (8).

This proof shows that $\sim t$ will suffice as our f. It is intuitively a compelling proof, given that you thing that t satisfies Def_t. Why would we think that? Well, we would think that if we thought $\to$ satisfies permutation, for in all logics from **C** upwards, Def_t holds. What we need is an argument for a conditional with permutation, *modus ponens*, and contraposition.

EXCURSUS: Interestingly, it is a fact that the popular systems that have a constant such as our f typically contain permutation, but this doesn't hold in general. There are formal systems that contain an f satisfying our conditions, without allowing permutation. For example, **LI** and **EW**. Furthermore, the connection between Def_t and permutation fails even in logics like those preferred by Priest and Sylvan.

This matrix shows that permutation is not valid in $\mathbf{DW}+((A \to \sim A) \vdash \sim A)+(A \leftrightarrow t \to A)+(A \wedge \sim A \to B \vee \sim B)$, which is frighteningly close, if not identical to Priest's logic Δ [113], with the addition of Def_t. So, the truth of Def_t is not sufficient for the validity of permutation, in *reasonable* logical systems. The model is given by:

a	$\sim a$
0	3
1	2
2	1
3	0

$\leq$	0	1	2	3
0	+	+	+	+
1	−	+	+	+
2	−	−	+	+
3	−	−	−	+

$\to$	0	1	2	3
0	3	3	3	3
1	1	2	2	3
2	0	1	2	3
3	0	0	1	3

$e = 2$

where conjunction and disjunction are generated by the order in the usual manner. Verifying that the algebra gives us a model of the logic in question is a matter of checking every postulate. If you wish to do that, go ahead. Better test one or two to get the flavour of it, and leave the rest to a computer. If neither of these appeal, rest assured that my computer has done it, and that I believe it. To falsify $p \to (q \to r) \vdash q \to (p \to r)$, set $p = 1$, $q = 3$, $r = 2$. (Thanks to John Slaney's `MaGIC` for generating this matrix faster than I ever could.) □

However, Priest has a point. For some conditionals, permutation is invalid, and this invalidity may explain the falsity of Def_t too. If the conditional has any modal force — if it evaluates across possible worlds — it will not satisfy permutation, and typically, it will not satisfy Def_t. However, non-modal conditionals seem legitimate in their own right: mathematical reasoning places no purchase on the order of premises, which can only be ignored if the conditional is non-modal, and permutation is satisfied. As an example, a mathematician may prove the Heine-Borel theorem, which states that:

Every closed and bounded subspace of the real line is compact.

by proving that if X is a closed subspace of the real line, then if X is bounded, it is compact. From this, she will have no qualms in deducing the conditional

If Y is closed, Y is compact.

from the premise that Y is a *bounded* subset of the real line. Juggling the order of premises is natural to mathematical reasoning. To reject this is a price the paraconsistent position must pay if it is to have uniformity. In addition, some kind of explanation will have to be given as to *why* premises cannot be permuted in mathematical contexts, other than its use in distinguishing the negation and the implication paradoxes.

It is open for the defender of the paraconsistent position to take a leaf out of some of the discussion from Chapter 2 to argue that mathematical reasoning uses permutation *enthymematically*. This means, permutation is taken to be invalid for the conditional in general, but it is assumed in mathematical contexts. This is quite plausible, but if this is done, and permutation holds for mathematical conditionals (those which have only mathematical propositions as atomic constituents) then it is not difficult to show that Def_t will hold for all mathematical propositions too, and so, at least Russell's paradox and its Curried version are shown to be of the same form. Then it may be argued that in contexts like naïve set abstraction, permutation will fail, just as both the paraconsistent and the paracomplete theorists take either excluded middle or *ex falso* to do. However, this is unmotivated. The paradoxes are intimately connected with contracting or expanding the set of premises used. Shuffling them around is not under a cloud in this context. If we want our reasoning in our naïve set theory to be as close as possible to classical mathematics, permutation should remain unscathed.

Perhaps a paraconsistent theorist can find an explanation of why permutation can fail in mathematical contexts, or that our argument fails at another point. If this is so, we are not yet finished. It is by no means clear that our concept of negation has only one sense — our natural language concepts are quite ambiguous. Our logical systems are designed to provide rigorous, formal analogues of our natural language arguments. The logical $\wedge$ in a particular system isn't meant to model or represent each of the natural language uses of the word "and," or each of the concepts of conjunction. Instead, it picks out a single sense of conjunction and models it. Similarly with $\sim$ in our systems. There is a particular sense of negation that receives formal treatment as $\sim$. There are other negation-like concepts (denial, failure, rejection, etc.) that we use, and there also are negation-like operators in our formal systems. Some of these operators are given in terms of implications. p implying something horrible is grounds for rejecting p, and if p is false, p does imply something horrible — itself. It is plain to see how the conditional has a connection with some kind of negation; which is a point Meyer and Martin labour [92].

To make this concrete, take $\perp$ to be a particular claim satisfying $\vdash \perp \to p$ for every single p. (Most theories have such an $\perp$, 'everything is true' or 'every set is an element of every other set' will do.) Take $-p$ to be $p \to \perp$. Given this, the liar and Russell paradoxes using this kind of negation *must* take advantage of the paracomplete solution. If it follows the paraconsistent solution, the paradoxical arguments conclude with $\perp$, which implies everything else.

If '$-$' is a negation, the paraconsistent solution is seen to be non-uniform, in that different kinds of negation paradoxes are treated differently. Now $-$ certainly looks like *some* kind of negation. It may not satisfy the law of excluded middle, but *ex falso*, contraposition, *modus tollens* and some de Morgan laws certainly hold. In any sensible logical system we have

$$\frac{p \to q}{-q \to -p} \qquad \frac{p \to q \quad -q}{-p} \qquad \frac{p \quad -p}{q} \qquad \begin{array}{l} -p \wedge -q \leftrightarrow -(p \vee q) \\ -p \vee -q \to -(p \wedge q) \end{array}$$

These certainly look like negation postulates of some kind or other. If we are willing to admit that this is a kind of negation, as it seems that we ought, then uniformity leads us to the conclusion that the paradoxes containing either kind of negation should be solved in the same way. We know that one must be solved by rejecting contraction — which is *reductio* for this negation and so the other must also. Not only this, but there are *more* negations where that one came from. If r is a rejected claim, $p \to r$ will act as a negation — it will satisfy each of the postulates given above, except *ex falso* (and even this will hold in the form $p, -p \vdash r$). It is a bit odd for all of *these* negation-like operators to use the paracomplete solution, and for one particular negation operator to differ.

So, uniformity gives us an equivocal result. For one, Priest has shown that the paraconsistent approach to the paradoxes has an underlying uniformity. However, we have just seen that invariably, there are a wide range of negation operators for which a paracomplete solution is the only available approach. If a general paracomplete solution can be shown to work for the remaining paradoxes, then uniformity points to paracompleteness, with in the context of contraction-free logics.

Expressibility. There is one important argument against a paracomplete account of the paradoxes. It seems that the analysis may fall foul to the *extended* paradoxes. This is because a naïve reading of the paracomplete solution would result in the view that the liar is neither true nor false. Let λ be a liar sentence: 'λ is not true'. In other words, we would have

$$\lambda \text{ is neither true nor false.}$$

(for we just said it). From this we can derive

$$\lambda \text{ is not true.}$$

which is λ itself. So, both λ and $\sim\lambda$ are true — which is what we wanted to avoid. What can we say in response?

One answer lies in the distinction between acceptance and denial, which has been championed by Terence Parsons [105]. The point relevant to us is that denying that p and asserting that $\sim p$ are different acts — and acts with differing contents. This can be seen to be so by way of the intuitionist, who denies the law of the excluded middle, and perhaps some instance of it. In other words, the intuitionist may deny $p \vee \sim p$ for some p. This does not commit her to accepting its negation, for a consequence of $\sim(p \vee \sim p)$ is (even according the intuitionist) $\sim p \wedge \sim\sim p$ which is not going to be accepted. Similarly, a paraconsistent logician may accept $p \wedge \sim p$ for some particular p. This is equivalent (according to many paraconsistent logicians) to $\sim(p \vee \sim p)$. However, it doesn't follow that the paraconsistent logician is in that breath denying $p \vee \sim p$. According to the paracomplete approach, the liar sentence is *denied*. It doesn't follow that its negation is accepted. This denial is expressed by saying "the liar sentence is not true," provided this is not construed as an assertion. This is either infelicitous, or it exploits an ambiguity in the word "not." "Not" is ambiguous between a denial and a negative assertion. It is not too hard to see how this ambiguity could arise — we often assert $\sim p$ in order to deny p, similarly, we often accept $\sim p$ on the grounds of the denial of p. Perhaps in this we play on the fact that this is appropriate for many propositions used in ordinary discourse. It does not follow that the distinction ought to be eliminated in *all* cases. Both the paraconsistent logician and the paracomplete reasoner agree on this point.

The distinction between these two acts ought to be maintained in our formal systems. This seems sensible — there needs to be some way to express our disapproval (or our theory's disapproval) of the paradoxical sentence, without getting ourselves (or our theories) tied into knots. This can be done. A theory has a set of sentences that are accepted, and those that are rejected. People have claims they accept, and those they reject. These need not be exhaustive (except for the bore who has an opinion on *everything*) but they had better be exclusive (except for the *really* confused). This is a way that the paracomplete solution can sidestep the extended paradoxes.

Another approach doesn't make use of the dichotomy between assertion and denial, but makes use of resources *within* our language to express truths about paradoxes. Instead of taking the paracomplete solution to conclude that the liar is neither true nor false, which we've seen lead us to undesirable conclusions, we can say that the liar is true if and only if it's false, and leave it at that. This is a way we can express the paradoxical nature of the liar, without falling into contradiction.

15.3 Conclusion

Where does this leave us with regard to negation? The primary point is one that has been continually echoed throughout this thesis: We need to do more research! We need more work on uniformly dealing with the paradoxes in a contraction-free way — in a way that treats genuine natural language negation analogously to the other negation-like operators defined in terms of implication and falsehoods. If this can be done (and there seems to be no indication that it cannot) then there is no compulsion forcing us to adopt a paraconsistent account of negation. If this is the case, we are not forced to accept the self-contradictory conclusions of paradoxes like the liar. We have seen that our logics give us the means to sidestep the extended paradoxes, so our approach is not self-defeating. The prospect for logics like our favourite logics is good indeed.

However, even if the favoured account of negation turns out to be radically different from the one we have considered — If, for example, excluded middle turns out to be true — then not all is lost for the project of contraction-free logics. Not at all. A modified account of negation coheres well with the positive systems we have studied. However we take negation to work, the story we have told about conjunction, disjunction and the conditional may remain. And if we wish our conditionals to satisfy *modus ponens* and other minimal principles, then contraction must go. So, our logics are useful in any case. Either way, logics without contraction are here to stay.

15.4 Note

[1] And for good reason too. This thesis was written over a period of three years, and in these three years, the author's views of negation have changed many a time.

Postscript

Logics without contraction are motivated by a number of independent sources. Firstly, the paradoxes. We have seen rich theories of arithmetic and truth which can be developed with contraction-free logics, while avoiding disaster at the hands of paradoxes of self-reference. We have also seen that using contraction-free logics we can regain a sensible set/class distinction within an expressive language. These theories are not 'crippled' by using a non-classical logic, instead, we have seen how classical insights can be retained, and enhanced, in the non-classical settings. We have seen results that show that the non-classical behaviour of paradoxical claims is restricted. Classical arithmetic and set/class theory can continue unscathed.

Contraction-free logics are not only useful for dealing with paradox. The second source of use for logics without contraction is the multiplicity of reasoning situations in which we must be 'resource aware'. In contexts of vagueness, logics of action, the 'logic' of sentence structure, and in situation theory, we must have conditionals which are sensitive to how antecedent assumptions are used. This requires a logic without contraction. These domains provide us with interpretations of the ternary relational semantics given by Routley and Meyer, and the operator semantics given by Fine. These semantic structures give us a whole range of possible applications of contraction-free formal systems, just as the possible-worlds semantics of modal logic opened up possibilities for interpreting classical modal logics.

So, contraction-free logics are not just *ad hoc* mechanisms designed in the light of philosophical conundrums. In this thesis we have seen that logics without contraction have semantic structures with meaningful interpretations, and that they have a well behaved proof theory. Contraction-free logics like ours are a natural part of the logical landscape, and they deserve to be studied for many years to come. With the many and fruitful lines of research uncovered within these pages, there will be much for future researchers to enjoy.

Bibliography

When you come,
bring the cloak that I left with Carpus at Troas,
also the books,
and above all, the parchments.
— PAUL THE APOSTLE 2 Timothy 4:13

ALAN ROSS ANDERSON AND NUEL D. BELNAP, JR.

[1] *Entailment: The Logic of Relevance and Necessity, Volume 1*, Princeton University Press, 1975.

[2] Section 8.11, the **P**−**W** problem.

[3] Section 28.2, propositional structures.

[4] Section 28.5, Gentzenisation of **R**.

ALAN ROSS ANDERSON, NUEL D. BELNAP, JR., AND J. MICHAEL DUNN.

[5] *Entailment: The Logic of Relevance and Necessity, Volume 2*, Princeton University Press, 1992.

[6] Section 50.1, situation semantics and first-degree entailment.

[7] Section 65, the undecidability of logics with contraction.

J. L. AUSTIN.

[8] "Truth," in *Proceedings of the Aristotelian Society* Supplementary **24** (1950). Reprinted in *Philosophical Papers* edited by J. O. Urmson and G. J. Warnock, Oxford University Press, 1961, pages 117–133.

JON BARWISE.

[9] "Constraints, channels, and the flow of information," Preprint, Philosophy Department, Indiana University. It is currently available as `Barwise-STA3.tex` by public `ftp` from `phil.indiana.edu`, 1993.

JON BARWISE AND JOHN ETCHEMENDY.

[10] *The Liar*, Oxford University Press, 1987.

JON BARWISE AND JOHN PERRY.

[11] *Situations and Attitudes*, MIT Press, Cambridge, Massachusetts, 1983.

[12] page 95,96, the inconsistency or incompleteness of some situations.

[13] "Shifting Situations and Shaken Attitudes: an interview with Barwise and Perry," *Linguistics and Philosophy* **8** (1985) 105–161. The quote is on page 118.

J. L. BELL AND A. B. SLOMSON.

[14] *Models and Ultraproducts: An Introduction*, North-Holland, Amsterdam, 1969. Chapters 3 and 4.

NUEL D. BELNAP, JR.

[15] "A Useful Four-Valued Logic," pages 8–37 in Dunn and Epstein [36], 1977.

Ross T. Brady.

[16] "Natural Deduction Systems for some Quantified Relevant Logics," *Logique et Analyse* **27** (1984) 355–377.

[17] "A Content Semantics for Quantified Relevant Logics I," *Studia Logica* **47** (1988) 111–127.

[18] "The Gentzenization and Decidability of **RW**" *Journal of Philosophical Logic* **19** (1990) 35–73.

[19] "The Gentzenization and Decidability of Some Contraction-less Relevant Logics" *Journal of Philosophical Logic* **20** (1991) 97–117.

Martin W. Bunder.

[20] "**BCK**-Predicate Logic as a Foundation For Multiset Theory," typescript, 1991. Faculty of Informatics, The University of Wollongong, Wollongong, New South Wales, 2500, Australia.

W. Buszkowski.

[21] "Compatibility of a Categorial Grammar with an Associated Category System," *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik* **31** (1982) 369–384.

Gennaro Cherchia, Barbara H. Partee and Raymond Turner.

[22] *Properties, Types and Meaning* Vol. I: Foundational Issues, Kluwer Academic Publishers, Dordrecht, 1989.

B. J. Copeland.

[23] "On When a Semantics is not a Semantics: Some Reasons for Disliking the Routley-Meyer Semantics for Relevance Logic," *Journal of Philosophical Logic* **8** (1979) 399–413.

The Count.

[24] 'Sesame Street,' a production of the Childrens' Television Workshop, 1993. The Count appears on almost every episode, and exclaims his love of counting in this fashion regularly.

Haskell B. Curry.

[25] "The Inconsistency of Certain Formal Logics," *Journal of Symbolic Logic* **7** (1942) 115–117.

Fred I. Dretske.

[26] *Knowledge and the Flow of Information*, Basil Blackwell, Oxford, 1981. The Xerox principle is introduced in page 57.

J. Michael Dunn.

[27] *The Algebra of Intensional Logics*, Doctoral Dissertation, University of Pittsburgh, Ann Arbor (University Microfilms) 1966.

[28] "A "Gentzen System" for Positive Relevant Implication," (abstract) *Journal of Symbolic Logic* **38** (1973) 356–357.

[29] "Intuitive Semantics for First-Degree Entailments and 'Coupled Trees'" *Philosophical Studies* **29** (1976) 149–168.

[30] "Relevance Logic and Entailment," pages 117–229 in Gabbay and Guenther [46], 1986.

[31] Section 2.2, the Lewis argument.

[32] Section 3.10, the Gentzenisation of **R**.

[33] Section 4.3, propositional structures for **R**.

[34] "Relevant Predication 1: the formal theory," *Journal of Philosophical Logic* **16** (1987) 347–381.

[35] "A Representation of Relation Algebras Using Routley-Meyer Frames," a paper presented to the 1992 Australasian Association for Logic Conference, 1992.

J. Michael Dunn and George Epstein, editors.

[36] *Modern Uses of Multiple-Valued Logics*, D. Reidel, Dordrecht, 1977.

J. Michael Dunn and Anil Gupta, editors.

[37] *Truth or Consequences*, Kluwer Academic Publishers, Dordrecht, 1990.

John Carew Eccles.

[38] *The Neurophysiological Basis of Mind: the principles of neurophysiology*, Clarendon Press, Oxford, 1953. The quotation is taken from page 1.

Euclid.

[39] *The Elements*, Books 1 to 6, translated by John Casey, Eleventh edition, Hodges, Figgis and Co., Dublin, 1892. Quote on page 12.

Solomon Feferman.

[40] "Towards Useful Type-Free Theories, I," *Journal of Symbolic Logic* **49** (1984) 75–111. The objection is on page 95.

Hartry Field.

[41] "Tarski's Theory of Truth," *Journal of Philosophy* **69** (1972) 347–375.

Kit Fine.

[42] "Models For Entailment," *Journal of Philosophical Logic* **3** (1974) 347–372.

[43] "Semantics for Quantified Relevance Logic," *Journal of Philosophical Logic* **17** (1988) 27–59.

[44] "Incompleteness for Quantified Relevance Logics," pages 205–255 in Norman and Sylvan [103] 1989.

Harvey Friedman and Robert K. Meyer

[45] "Whither Relevant Arithmetic?" *Journal of Symbolic Logic* **57** (1992) 824–831.

D. Gabbay and F. Guenther.

[46] *The Handbook of Philosophical Logic*. Volume III: Alternatives to Classical Logic, D. Reidel, Dordrecht, 1986.

P. T. Geach.

[47] "On Insolubilia" *Analysis* **15** (1955) 71–72.

Steve Giambrone

[48] "$\mathbf{TW}^{+}$ and $\mathbf{RW}^{+}$ are Decidable," *Journal of Philosophical Logic* **14** (1985) 235–254.

Steve Giambrone and Robert K. Meyer.

[49] "Completeness and Conservative Extension Results for Some Boolean Relevant Logics," *Studia Logica* **48** (1989) 1–14.

Edward Gibbon.

[50] *The Decline and Fall of the Roman Empire*, volume 4. Edited by Olifant Smeaton, Everyman's Edition, J. M. Dent and Sons, London, 1910. The quotation is from Chapter XLIV, on page 400 of the 1910 edition.

Jean-Yves Girard.

[51] "Linear Logic," *Theoretical Computer Science* **50** (1987) 1–101.

[52] "The Geometry of Interaction 1: interpretation of system F," *Logic Colloquium '88* R. Ferro, C. Bonott, S. Valentini and A. Zanardo eds., North Holland, Amsterdam 1989, pages 221–260.

Graham Greene.

[53] *The Quiet American* William Heinemann and the Bodley Head, London, 1973 (First published 1955.) The quotation is taken from page 196 of the 1973 edition.

Anil Gupta.

[54] "Modal Logic and Truth," *Journal of Philosophical Logic* **7** (1978) 441–472.

Susan Haack.

[55] *Deviant Logic* Cambridge University Press, Cambridge 1974.

[56] Page 26, logic as a theory on a par with other scientific theories.

Louise Schmir Hay.

[57] "Axiomatisation of the Infinite-Valued Predicate Calculus," *Journal of Symbolic Logic* **28** (1963) 77–86.

Rob Hirst.

[58] 'My Country,' on *Earth and Sun and Moon* by Midnight Oil, Columbia Records 473605 2, 1993. Track published by Warner, Chappell Music, Sprint Music.

B. Jónsson and A. Tarski

[59] "Representation problems for relation algebras" *Bulletin of the American Mathematical Society* **54** (1948) 1192.

Michael Jubien.

[60] "On Properties and Property Theory," pages 159–175 in Cherchia, Partee and Turner [22], 1989. The quote in on page 162–163.

Makoto Kanazawa.

[61] "The Lambek Calculus Enriched with Additional Connectives," *Journal of Logic, Language and Information* **1** (1992) 141–171.

Saul Kripke.

[62] "The Problem of Entailment," (abstract) *Journal of Symbolic Logic* **24** (1959) 324.

[63] "Outline of a Theory of Truth," *Journal of Philosophy* **72** (1975) 690–716.

Mark Lance and Philip Kremer.

[64] "The Logical Structure of Linguistic Commitment II: systems of relevant commitment entailment," presented to the 1992 Australasian Association for Logic Conference, Canberra.

Hugues Leblanc, editor.

[65] *Truth Syntax and Modality*, Proceedings of the Temple University Conference on Alternative Semantics, North-Holland, Amsterdam, 1973.

E. J. Lemmon.

[66] *Beginning Logic*, Van Nostrad Reinhold, London, 1965.

John Lennon and Paul McCartney.

[67] 'With a Little Help from my Friends,' on *Sgt. Pepper's Lonely Hearts Club Band* by the Beatles, EMI Records CDP7 45442 2, 1967, 1987 (Digital Remastering). Track published by Northern Songs Ltd. The track is sung by Ringo Starr.

Clarence Irving Lewis and Cooper Harold Langford.

[68] *Symbolic Logic*, The Century Co., New York and London, 1932. Second Edition, Dover, New York, 1959. The argument is on page 250.

M. H. Löb.

[69] "Solution of a Problem of Leon Henkin," *Journal of Symbolic Logic* **20** (1955) 115–118.

Jan Łukasiewicz.

[70] "Philosophical Remarks on Many-Valued Systems," pages 153–178 in Łukasiewicz [73], 1970. (English Translation of "Philosophische Bemerkungen zu mehrwertigen Systemen des Aussagenkalkülis," *Comptes rendus des séances de la Société des Sciences et des Lettres de Varsovie* **23** (1930) 51–77.)

[71] Page 174 (of the English translation) apagogic inference schemata.

[72] "In Defence of Logistic," pages 236–249 in Łukasiewicz [73], 1970. (English Translation of "W obronie logistyki," *Studia Gnesnensia* **15** (1937).) Quote on page 249.

[73] *Selected Works*, North Holland, Amsterdam, 1970. Edited by L. Borkowski.

R. C. Lyndon.

[74] "The Representation of Relational Algebras," *Ann. of Math. Series 2* **51** (1950) 707–729.

Penelope Maddy.

[75] "Proper Classes," *Journal of Symbolic Logic* **48** (1983) 113–139.

MAO TSE-TUNG.

[76] *On the Correct Handling of Contradictions Among the People* February 1957. Quote and reference taken from *Quotations of Chairman Mao Tse-Tung* edited by Stuart R. Schram, Frederick A. Praeger Publishing, New York, 1968, on page 174.

EDWIN MARES.

[77] "Semantics for Relevance Logic with Identity," *Studia Logica* **51** (1992) 1–20.

ERROL P. MARTIN.

[78] *The* $\mathbf{P}-\mathbf{W}$ *Problem* Doctoral Dissertation, Australian National University, Canberra, Australia, 1978.

ERROL P. MARTIN AND ROBERT K. MEYER.

[79] "Solution to the $\mathbf{P}-\mathbf{W}$ Problem," *Journal of Symbolic Logic* **47** (1982) 869–886.

ERROL P. MARTIN AND JOHN K. SLANEY.

[80] *Contraction Free Logics*, unpublished and unfinished manuscript. Automated Reasoning Project, Australian National University, 199?, Chapter 3.

KARL MARX.

[81] *Selected Writings*, edited by David McLellan, Oxford University Press, 1977. The quote is on page 487, and it is taken from *Capital*.

VANN MCGEE.

[82] *Truth, Vagueness and Paradox: An Essay on the Logic of Truth*, Hackett Publishing Company, Indianapolis, 1991.

CHRISTOPHER MENZEL.

[83] "Actualism, Ontological Commitment, and Possible World Semantics," *Synthese* **85** (1990) 355–389.

ROBERT K. MEYER.

[84] "Entailment" *Journal of Philosophy* **68** (1971) 808–818.

[85] "A General Gentzen System for Positive Implication Calculi," *Relevance Logic Newsletter* **1** (1976) 189–201.

[86] "Intuitionism, Entailment, Negation," pages 168–198 in Leblanc [65], 1973.

[87] "Metacompleteness," *Notre Dame Journal of Formal Logic* **17** (1976) 501–516.

[88] "Conserving Positive Logics," *Notre Dame Journal of Formal Logic* **14** (1973) 224–236.

[89] "Relevant Arithmetic," (abstract) *Bulletin of the Section of Logic of the Polish Academy of Sciences* **5** (1976) 133–137.

[90] "Arithmetic Relevantly Formulated," typescript. Automated Reasoning Project, Australian National University, 197?.

ROBERT K. MEYER, STEVE GIAMBRONE AND ROSS T. BRADY.

[91] "Where Gamma Fails," *Studia Logica* **43** (1984) 247–256.

ROBERT K. MEYER AND ERROL P. MARTIN

[92] "Logic on the Australian Plan," *Journal of Philosophical Logic* **15** (1986) 305–332.

Robert K. Meyer and Chris Mortensen.

[93] "Inconsistent Models of Relevant Arithmetics," *Journal of Symbolic Logic* **49** (1984) 917–929.

Robert K. Meyer and Richard Routley.

[94] "Algebraic Analysis of Entailment I," *Logique et Analyse* **15** (1972) 407–428.

Robert K. Meyer, Richard Routley and J. Michael Dunn.

[95] "Curry's Paradox," *Analysis* **39** (1979) 124–128.

Robert K. Meyer and John K. Slaney.

[96] "Abelian Logic (from A to Z)," pages 245–238 in Priest, Routley and Norman [117], 1989.

Moh Shaw-Kwei.

[97] "Logical Paradoxes for Many-Valued Systems," *Journal of Symbolic Logic* **19** (1954) 37–40.

Augustus de Morgan.

[98] "On the Syllogism, no IV, and on the logic of relations" *Transactions of the Cambridge Philosophical Society* **10** (1860) 331–358.

Chris Mortensen.

[99] "Inconsistent Number Systems," *Notre Dame Journal of Formal Logic* **29** (1988) 45–60.

[100] "Anything is Possible," *Erkenntnis* **30** (1989) 3319–337.

John Myhill.

[101] "Paradoxes," *Synthese* **60** (1984) 129–143.

[102] "Real Implication," pages 157–166 in Norman and Sylvan [103], 1989.

Jean Norman and Richard Sylvan, editors.

[103] *Directions in Relevant Logic*, Kluwer Academic Publishers, Dordrecht, 1989.

Richard T. Oehrle.

[104] "Dynamic Categorial Grammar," pages 79–128 in *Formal Grammar: Theory and Implementation*, edited by Robert Levine, Oxford University Press, New York, 1992.

Terence Parsons.

[105] "Assertion, Denial, and the Liar Paradox," *Journal of Philosophical Logic* **13** (1984) 137–152.

[106] "True Contradictions," *Canadian Journal of Philosophy* **20** (1990) 335–354.

Christopher Peacocke.

[107] "Necessity and Truth Theories," *Journal of Philosophical Logic* **7** (1978) 473–500.

Charles Sanders Peirce.

[108] "Algebra of Logic" *American Journal of Mathematics* **3** (1880) 15–57. Reprinted in [110] volume 3, 104–157.

[109] "Description of a notation for the logic of relatives, resulting from an amplification of the conceptions of Boole's calculus of logic." pages 27–98 in [110] volume 3.

[110] *Collected Papers*, The Belknap Press of Harvard University Press, 1960. Volumes 1–4, Edited by C. Hartshorne and P. Weiss.

Stephen Pollard.

[111] *Philosophical Introduction to Set Theory*, University of Notre Dame Press, Notre Dame, Ind. 1990. The quotation is taken from page 40.

Graham Priest.

[112] "Sense, Entailment and *Modus Ponens*" *Journal of Philosophical Logic* **9** (1980) 415–435. Comment on page 433

[113] *In Contradiction: A Study of the Transconsistent*, Martinus Nijhoff, The Hague, 1987.

[113.5] Section 1.8, Berry's Paradox.

[114] "Classical Logic Aufgehoben," pages 131–148 in Priest, Routley and Norman [117], 1989.

[115] "Intensional Paradoxes," *Notre Dame Journal of Formal Logic* **32** (1991) 193–221.

[116] "The Structure of the Paradoxes of Self-Reference," Read to the Australasian Association for Logic Conference, Sydney, 1991.

Graham Priest, Richard Routley and Jean Norman, editors.

[117] *Paraconsistent Logic: Essays on the Inconsistent*, Philosophia Verlag, München, 1989.

Graham Priest and Richard Sylvan.

[118] "Simplified Semantics for Basic Relevant Logics," *Journal of Philosophical Logic* **21** (1992) 217–232.

Arthur Prior.

[119] "Curry's Paradox and 3-valued Logic", *Australasian Journal of Philosophy* **33** (1955) 177–182.

Willard Van Orman Quine.

[120] *Philosophy of Logic*, Prentice Hall, Englewood Cliffs, 1970. Quote on page 81.

Frank P. Ramsey.

[121] "The Foundations of Mathematics," *Proceedings of the London Mathematical Society* **25** (1925) 338–384; Reprinted in [122], 1978.

[122] *Foundations: Essays in Philosophy, Logic, Mathematics and Economics* Collected Essays. Routledge and Kegan Paul, 1978. Edited by D. H. Mellor

Greg Restall.

[123] "Deviant Logic and the Paradoxes of Self Reference," to appear in *Philosophical Studies* **70** (1993) 279–303.

[124] "Simplified Semantics for Relevant Logics (and some of their Rivals)," *Journal of Philosophical Logic* **22** (1993) 481–511.

[125] "Comparing Deviant Logics," presented to the Australian Association for Logic Conference, Canberra 1992.

[126] "How to be *Really* Contraction Free," *Studia Logica* **52** (1993) 381–391.

[127] "Contraction, Modalities, Relevance," to appear in *Logique et Analyse* (199?).

RICHARD ROUTLEY AND ROBERT K. MEYER.

[128] "Semantics of Entailment 2," *Journal of Philosophical Logic* **1** (1972) 53–73.

[129] "Semantics of Entailment 3," *Journal of Philosophical Logic* **1** (1972) 192–208.

[130] "Semantics of Entailment 1," pages 199–243 in Leblanc [65], 1973.

RICHARD ROUTLEY WITH VAL PLUMWOOD, ROBERT K. MEYER AND ROSS T. BRADY.

[131] *Relevant Logics and their Rivals*, Ridgeview, 1982.

[132] Section 4.1, extensions of basic relevant logics.

[133] Section 4.3, modelling conditions for the extensions.

[134] Section 4.6, completeness proofs.

BERTRAND RUSSELL.

[135] "The Logic of Relations," *Rivista di Matematica* **7** (1900) 115–148.

[136] "On Denoting," *Mind* new series **14** (1905) 479–493. Quote on pages 484 and 485.

[137] "On Some Difficulties in the Theory of Transfinite Numbers and Order Types," *Proceedings of the London Mathematical Society* (Series 2) **4** (1905) 29–53.

HAROLD SCHELLINX.

[138] "Some Syntactical Observations on Linear Logic," *Journal of Logic and Computation* **1** (1991) 537–559.

E. F. SCHUMACHER.

[139] *Small is Beautiful: a study of economics as if people mattered*, Blond and Briggs, London, 1973.

E. SCHRÖDER.

[140] *Vorlesungen über die Algebra der Logik (Exakte Logic). Dritte Band: Algebra und Logik der Relative* B. G. Tuebner, Leipzig, 1895.

WILLIAM SHAKESPEARE.

[141] *All's Well that Ends Well*, 1601. The edition used was 'The Arden Shakespeare' Third Edition, edited by G. K. Hunter, Methuen and Co., Strand, 1959. The quotation is from Act 1, Scene ii, lines 74 and 75, on page 20 of the 1959 edition.

PAUL SIMON.

[142] 'Proof,' on *The Rhythm of the Saints* by Paul Simon, Warner Brothers Records, Inc., 759926098-4, 1990. Track published by Warner Brothers.

JOHN K. SLANEY

[143] "A Metacompleteness Theorem for Contraction-free Relevant Logics," *Studia Logica* **43** (1984) 159–168. Comment on page 167.

[144] "Vagueness Revisited," Australian National University, Automated Reasoning Project Technical Report, TR-ARP-15/88, 1988.

[145] Page 5, quote on grass.

[146] Page 37, quote on decidability.

[147] "**RWX** is not Curry Paraconsistent," pages 472–480 in Priest, Routley and Norman [117], 1989.

[148] "A Solution to a Problem of Ono and Komori," *Journal of Philosophical Logic* **18** (1989) 103–111.

[149] "The Square Root of 2 is Irrational (and no funny business)," typescript 197?. Automated Reasoning Project, Australian National University.

[150] "Finite Models for some Non-Classical Logics," Australian National University, Automated Reasoning Project Technical Report, TR-ARP 2/90, 1990.

[151] "A General Logic," *Australasian Journal of Philosophy* **68** (1990) 74–88.

[152] "The Implications of Paraconsistency," Proceedings of IJCAI-91, Vol. 2 (1991) 1052–1057.

[153] "A Simple Proof of Consistency for a Naive Axiom of Propositional Comprehension," *Journal of Non Classical Logic* **4** (1987) 23–31.

JOHN K. SLANEY AND GUSTAV MEGLICKI

[154] "MaGIC v. 2.0 Notes and Guide," Australian National University, Automated Reasoning Project Technical Report, TR-ARP 10/89, 1990.

JOHN K. SLANEY, TIMOTHY SURENDONK AND RODERIC GIRLE.

[155] "Time, Truth and Logic," Australian National University, Automated Reasoning Project Technical Report, TR-ARP-11/89, 1989.

ROBERT P. SWIERENGA.

[156] "Clio and Computers: a survey of computerised research in history," *Computers and the Humanities* **5** (1970) 1–??. The quotation is on page 1.

RICHARD SYLVAN (see also RICHARD ROUTLEY).

[157] "Ubiquitous Vagueness Without Embarrassment," typescript, 1992. Philosophy Department, Research School of Social Sciences, Australian National University.

A. TARKSI.

[158] "On the Calculus of Relations," *Journal of Symbolic Logic* **6** (1941) 73–89.

[159] "Some metalogical results concerning the calculus of relations," Abstract *Journal of Symbolic Logic* **18** (1953) 188–189.

P. B. THISLEWAITE, M. A. MCROBBIE, AND R. K. MEYER.

[160] *Automated Theorem-Proving in Non-Classical Logics*, Pitman, London; Wiley, New York and Toronto, 1988.

RICHMOND H. THOMASON.

[161] "Motivating Ramified Type Theory," pages 47–62 in Cherchia, Partee and Turner [22], 1989.

ALASDAIR URQUHART.

[162] "Semantics for Relevance Logics," *Journal of Symbolic Logic* **37** (1972) 159–169.

[163] "The Undecidability of Entailment and Relevant Implication," *Journal of Symbolic Logic* **49** (1984) 1059–1073.

[164] "The Complexity of Decision Procedures in Relevance Logic," pages 61–76 in Dunn and Gupta [37], 1990.

Linda Wetzel.

[165] "What Are Occurrences of Expressions?" *Journal of Philosophical Logic* **22** (1993) 215–219.

Richard Whately.

[166] *Elements of Logic,* ninth edition, Longmans, Green and Co., London, 1913. The quotation is from page 65.

Richard B. White.

[167] "The Consistency of the Axiom of Comprehension in the Infinite-Valued Predicate Logic of Łukasiewicz," *Journal of Philosophical Logic* **8** (1979) 509–534.

Robert Wolf.

[168] "Are Relevant Logics Deviant?" *Philosophia* **7** (1977) 327–340. Quote from page 336–337.

Monsieur Hevelius from Dantzick.

[169] Letter to the Publisher, *Philosophical Transactions of the Royal Society of London* **7** (1672) 4017–4018. The letter was translated from Latin into English by the publisher.

Archbishop Edwin Sandys.

[170] *Sermons and Miscellaneous Pieces*, Cambridge University Press, 1841. The quotation is taken from pages 50 and 51.

G. E. Minc.

[171] "Cut-Elimination Theorem in Relevant Logics (Russian)," *Isslédoviniá po komstruktivnoj matematiké i matematičesok logike V* ed. J. V Matijasevic and O. A. Slisenko, Moscow, 1972, 90–97. English translation on *The Journal of Soviet Mathematics* **6** (1976) 422–428.

Ross T. Brady.

[172] "The Simple Consistency of a Set Theory Based on the Logic **CSQ**," *Notre Dame Journal of Formal Logic* **21** (1983) 431–439.

Albert Visser

[173] "Four Valued Semantics and the Liar," *Journal of Philosophical Logic* **13** (1984) 181–212.

Arthur Pap

[174] *Semantics and Necessary Truth*, Yale University Press, 1958.

Candidate's Publications

P.1 Papers Published

⟨1⟩ "A Note on Naïve Set Theory in **LP**," *Notre Dame Journal of Formal Logic* **33** (1992) 422–432.

⟨2⟩ "Deviant Logic and the Paradoxes of Self Reference," *Philosophical Studies* **70** (1993) 279–303.

⟨3⟩ "Simplified Semantics for Relevant Logics (and some of their rivals)," *Journal of Philosophical Logic* **22** (1993) 481–511.

⟨4⟩ "How to be *Really* Contraction Free," *Studia Logica* **52** (1993) 381–391.

P.2 Papers Accepted for Publication

⟨5⟩ "Contraction, Modalities, Relevance," to appear in *Logique et Analyse*.

⟨6⟩ "Four-Valued Semantics for Relevant Logics (and some of their rivals)," to appear in *Journal of Philosophical Logic*.

⟨7⟩ "Subintuitionistic Logics," to appear in *Notre Dame Journal of Formal Logic.*

P.3 Abstracts, Reviews, Encyclopedia Entries

⟨8⟩ "How to be really contraction free," (Abstract) *Journal of Symbolic Logic* **57** (1992) 1501-1502.

⟨9⟩ "Comparing Deviant Logics," (Abstract) to appear in *Journal of Symbolic Logic*.

⟨10⟩ "Logical Laws," an entry for Routledge's forthcoming *Encyclopedia of Philosophy*

⟨11⟩ "Critical Notice of John P. Cleave's *A Study of Logics* forthcoming in *Modern Logic*.