ROUTES TO TRIVIALITY

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Abstract: It is known that a number of inference principles can be used to trivialise the axioms of naïve comprehension — the axioms underlying the naïve theory of sets. In this paper we systematise and extend these known results, to provide a number of general classes of axioms responsible for trivialising naïve comprehension.

Keywords: Abelian logic, contraction, Curry's paradox, naïve comprehension, Non-classical implicational logics, Peirce's law.

1 Introduction

In this paper we consider logics with a language containing infinitely many propositional variables, the first two of which we shall call p and q, at least the binary connective $\rightarrow$, and which are closed under *modus ponens* (MP) and uniform substitution of arbitrary variables. We envisage theories that are extensions of given logics obtained by the addition of, at least, the naïve comprehension principles (NC$\rightarrow$) and (NC$\leftarrow$), which we introduce below.

2 Curry's Paradox

Curry [2] showed that any logic closed under *modus ponens*, uniform substitution and contraction (W) would trivialise the naïve comprehension principles (NC$\rightarrow$) and (NC$\leftarrow$):

(NC$\rightarrow$) $\quad a \in \{x : \varphi(x)\} \rightarrow \varphi(a)$
(NC$\leftarrow$) $\quad \varphi(a) \rightarrow a \in \{x : \varphi(x)\}$

Choose $\varphi(x)$ to be $x \in x \rightarrow p$, and let a be $\{x : \varphi(x)\}$. Then as instances of (NC$\rightarrow$) and (NC$\leftarrow$) we have $a \in a \rightarrow (a \in a \rightarrow p)$ and $(a \in a \rightarrow p) \rightarrow a \in a$.

Let us suppose that W (contraction) is an axiom of the logic:

W $\quad (p \rightarrow (p \rightarrow q)) \rightarrow (p \rightarrow q)$

Then we have the following proof:

(1)	$a \in a \to (a \in a \to p)$	NC$\to$
(2)	$(a \in a \to (a \in a \to p)) \to (a \in a \to p)$	instance of W
(3)	$a \in a \to p$	$1, 2$ MP
(4)	$(a \in a \to p) \to a \in a$	NC$\leftarrow$
(5)	$a \in a$	$3, 4$ MP
(6)	p	$3, 5$ MP

But p is arbitrary. Other than *modus ponens* the only other principle required for this proof is that of contraction; so any logic with (NC$\to$), (NC$\leftarrow$) and W is trivial.

However, contraction is not the only route to triviality. There are logics which are contraction free that still trivialise naïve comprehension (NC) (comprising both principles (NC$\to$) and (NC$\leftarrow$)). See [6] and [7].

It is worth noting that the study of naïve comprehension has respectable mathematical applications. For example, in some recent work of Terui, naïve set theory (without extensionality) based on linear logic or on affine logic (BCK) is used to obtain results on complexity [9]. These applications are not the subject of this paper, however, this paper will help researchers delineate the kinds of logics that will provide an appropriate context for the study of naïve comprehension principles.

3 Other known offenders

3.1 Abelian Logic, with the Axiom of Relativity

Abelian logic [4] has the unusual Axiom of Relativity, which we have dubbed A.

$$\mathsf{A} \quad ((p \to q) \to q) \to p$$

A also trivialises (NC). As before, let $a = \{x : \varphi(x)\}$ and this time choose $\varphi(x) = p \to x \in x$. Then as instances of (NC$\to$) and (NC$\leftarrow$) we have $a \in a \to (p \to a \in a)$ and $(p \to a \in a) \to a \in a$.

(1)	$(p \to a \in a) \to a \in a$	NC$\leftarrow$
(2)	$((p \to a \in a) \to a \in a) \to p$	instance of A
(3)	p	$1, 2$ MP

Although A is not a contraction axiom itself, a contracting implication is definable in the full language of Abelian Logic [6]. There is, however, no contracting implication definable in the implicational fragment of Abelian logic. (This fact follows from the main result in [7].)

3.2 A 3-Valued Logic with SC and L

Consider the implicational logic $\mathbf{L_3}$ given by the following table:

$\to$	0	1	2
0^*	0	2	1
1	1	0	2
2	2	1	0

The tautologies of $\mathbf{L_3}$ are those formulas that always receive the designated value, 0, under any interpretation. Amongst the theorems of this logic, we draw attention here to two, each of which trivialises naïve comprehension [7]. The first, L, is as follows:

$$\mathsf{L} \quad (p \to (p \to q)) \to (((p \to q) \to p) \to q)$$

Let $a = \{x : \varphi(x)\}$, and choose $\varphi(x)$ to be $x \in x \to p$ and substitute a for x throughout. As instances of (NC$\to$) and (NC$\leftarrow$) we have $a \in a \to (a \in a \to p)$ and $(a \in a \to p) \to a \in a$

(1)	$a \in a \to (a \in a \to p)$	NC$\to$
(2)	$(a \in a \to (a \in a \to p)) \to (((a \in a \to p) \to a \in a) \to p)$	L
(3)	$((a \in a \to p) \to a \in a) \to p$	1, 2 MP
(4)	$(a \in a \to p) \to a \in a$	NC$\leftarrow$
(5)	p	3, 4 MP

The second, SC (super contraction) is as follows:

$$\mathsf{SC} \quad (p \to (p \to q)) \to q$$

As before, with the appropriate substitutions we have $a \in a \to (a \in a \to p)$ as an instance of (NC$\to$). Then the proof of triviality goes like this:

(1)	$a \in a \to (a \in a \to p)$	NC$\to$
(2)	$(a \in a \to (a \in a \to p)) \to p$	SC
(3)	p	1, 2 MP

These are important trivialising axioms for two reasons: First, they do not reduce to contraction in any sense; no implication satisfying (W) is definable in this logic [7]. Second, with SC, (NC$\to$) is trivial on its own. We did not appeal to (NC$\leftarrow$) at any stage.

3.3 Peirce's Law (PL)

Martin Bunder [1] points out that Peirce's Law (PL) will trivialise (NC).

$$\mathsf{PL} \quad ((p \to q) \to p) \to p$$

As before, with the appropriate substitutions we have $a \in a \to (a \in a \to p)$ and $(a \in a \to p) \to a \in a$ as instances of (NC$\to$) and (NC$\leftarrow$).

(1)	$(a \in a \to p) \to a \in a$	NC $\leftarrow$
(2)	$((a \in a \to p) \to a \in a) \to a \in a$	instance of PL
(3)	$a \in a$	1, 2 MP
(4)	$a \in a \to (a \in a \to p)$	NC $\to$
(5)	$a \in a \to p$	3, 4 MP
(6)	p	3, 5 MP

Each of the axioms responsible for trivialising (NC) is purely implicational, and so the implicational fragments of logics including any of these axioms, such as, classical, intuitionistic, relevant and Abelian logics, all trivialise (NC).

4 Generalisations

This paper: (i) asks what is special about these logics that allows them to trivialise (NC), and (ii) tries to express these properties in more generalised forms. When trying to generalise the properties of these logics leading to triviality with (NC), we first notice that each of the above axioms can be generalised, to produce classes of formulas responsible for trivialising (NC). We will start with Moh Shaw-Kwei's result concerning generalisations of contraction. Before we can do that, however, we need to introduce two small notational devices to simplify the presentation of nested conditionals. First, for conditionals with multiple antecedents, we will use a traditional notation: Set $A \;{}_{[0]}\!\!\to B$ to be B, and $A \;{}_{[n+1]}\!\!\to B$ to be $A \to (A \;{}_{[n]}\!\!\to B)$. We will also need to consider conditionals with nested *consequents*, so consider this notation: Set $A \to_{[0]} B$ to be A, and $A \to_{[n+1]} B$ to be $(A \to_{[n]} B) \to B$. With this notation in hand, let us consider generalisations of the axioms trivialising (NC).

4.1 Contraction (W)

We can think of the axiom (W) a "$2 \rightsquigarrow 1$ contraction", as it has the form $(A \;{}_{[2]}\!\!\to B) \to (A \;{}_{[1]}\!\!\to B)$. This terminology is due to Ono [3]. Moh Shaw-Kwei [8] has shown that any $(n+1) \rightsquigarrow n$ contracting implication trivialises (NC).

$$\mathsf{W}_{n+1 \rightsquigarrow n} \qquad (p \to_{[n+1]} q) \to (p \to_{[n]} q)$$

Choose $\varphi(x)$ to be $x \in x \;{}_{[n]}\!\!\to p$. The appropriate instances of (NC$\to$) and (NC$\leftarrow$) are then $a \in a \to (a \to a \;{}_{[n]}\!\!\to p)$ and $(a \in a \;{}_{[n]}\!\!\to p) \to a \in a$. The proof of triviality generalises the case for $2 \rightsquigarrow 1$ contraction.

(1)	$a \in a \rightarrow (a \in a \rightarrow_{[n]} p)$	NC$\rightarrow$
(2)	$(a \in a \rightarrow (a \in a \rightarrow_{[n]} p)) \rightarrow (a \in a \rightarrow_{[n]} p)$	$(n+1 \rightsquigarrow n)$ W
(3)	$a \in a \rightarrow_{[n]} p$	$1, 2$ MP
(4)	$(a \in a \rightarrow_{[n]} p) \rightarrow a \in a$	NC$\leftarrow$
(5)	$a \in a$	$3, 4$ MP
$\vdots$	$\vdots$	$\vdots$
$(6+n)$	p	$3, 5$ MP

4.2 Axiom of Relativity (A)

The axiom A can be extended to contain k occurrences of q, where $k \geq 2$.

$$\mathsf{A}_k \quad (p \rightarrow_{[k]} q) \rightarrow p$$

The triviality proof of A_k (for $k \geq 2$) uses $p \rightarrow_{[k-1]} x \in x$ for $\varphi(x)$.

(1)	$(p \rightarrow_{[k-1]} a \in a) \rightarrow a \in a$	NC$\leftarrow$
(2)	$((p \rightarrow_{[k-1]} a \in a) \rightarrow a \in a) \rightarrow p$	instance of A_k
(3)	p	$1, 2$ MP

Notice that these proofs are simpler than the case for $(n+1 \rightsquigarrow n)$ contraction, as here only one instance of *modus ponens* is required.

4.3 Super Contraction (SC)

We thought of (SC) as $2 \rightsquigarrow 0$ contraction. We now consider $k \rightsquigarrow 0$ contraction, with k occurrences of the contracted formula.

$$\mathsf{SC}_k \quad (p \rightarrow_{[k]} q) \rightarrow q$$

For the triviality proof, choose $x \in x \rightarrow_{[k-1]} p$ for $\varphi(x)$. The appropriate instances of (NC$\rightarrow$) and (NC$\leftarrow$) are then $a \in a \rightarrow (a \in a \rightarrow_{[k-1]} p)$ and $(a \in a \rightarrow_{[k-1]} p) \rightarrow a \in a$.

(1)	$a \in a \rightarrow (a \in a \rightarrow_{[k-1]} p)$	NC$\leftarrow$
(2)	$(a \in a \rightarrow_{[k]} p) \rightarrow p$	SC_k
(3)	p	$1, 2$ MP

Each SC_k thus trivialises (NC).

4.4 (L)

(L): $(p \rightarrow (p \rightarrow q)) \rightarrow (((p \rightarrow q) \rightarrow p) \rightarrow q)$ can also be rearranged and extended. It can be extended by adding antecedents of either form $p \rightarrow (p \rightarrow q)$ or $(p \rightarrow q) \rightarrow p$. The proof of triviality proceeds by choosing the appropriate direction of (NC) and applying MP as many times as necessary. Rearrangements of (L), such as those obtained by swapping the order of $p \to (p \to q)$ and $(p \to q) \to p$, or by deleting one of them, or by replacing one with the other also trivialise (NC). For example, if we add $(p \to q) \to p$ as an antecedent to (L) we get the formula

$$\mathsf{L}' \quad ((p \to q) \to p) \to ((p \to (p \to q)) \to (((p \to q) \to p) \to q))$$

Using the same choice of $a = \{x : x \in x \to p\}$ as in the triviality proof for (L), we have as instances of (NC$\to$) and (NC$\leftarrow$) $a \in a \to (a \in a \to p)$ and $(a \in a \to p) \to a \in a$. The proof of triviality then follows:

(1)	$(a \in a \to p) \to a \in a$	NC$\leftarrow$
(2)	$((a \in a \to p) \to a \in a) \to ((a \in a \to (a \in a \to p)) \to$ $(((a \in a \to p) \to a \in a) \to p))$	L'
(3)	$(a \in a \to (a \in a \to p)) \to (((a \in a \to p) \to a \in a) \to p)$	$1, 2$ MP

(3) is an instance of (L) and the proof then proceeds as for (3.2).

Consider a *rearrangement* of (L) where we permute the antecedents.

$$\mathsf{L}'' \quad ((p \to q) \to p) \to ((p \to (p \to q)) \to q)$$

The triviality proof is as in (3.2), except that the two applications of *modus ponens* are reversed.

(1)	$((a \in a \to p) \to a \in a) \to ((a \in a \to (a \in a \to p)) \to p)$	L''
(2)	$(a \in a \to p) \to a \in a$	NC $\leftarrow$
(3)	$(a \in a \to (a \in a \to p)) \to p$	$1, 2$ MP
(4)	$a \in a \to (a \in a \to p)$	NC$\to$
(5)	p	$3, 4$ MP

4.5 Peirce's Law (PL)

(PL): $((p \to q) \to p) \to p$ can be extended in the same way as (L); that is, by adding antecedents of either $p \to (p \to q)$ or $(p \to q) \to p$ in what ever quantity and order we choose. For example: consider the case in which we add an antecedent of the form $p \to (p \to q)$. Our new axiom is then

$$\mathsf{PL}' \quad (p \to (p \to q)) \to (((p \to q) \to p) \to p)$$

We choose a as before, and again, we have as instances of (NC$\to$) and (NC$\leftarrow$), $a \in a \to (a \in a \to p)$ and $(a \in a \to p) \to a \in a$. The proof follows:

(1)	$(a \in a \to (a \in a \to p)) \to (((a \in a \to p) \to a \in a) \to a \in a)$	PL'
(2)	$a \in a \to (a \in a \to p)$	NC $\to$
(3)	$((a \in a \to p) \to a \in a) \to a \in a$	MP

(3) is an instance of (PL) and the proof now proceeds as for (3.3).

5 Three classes of formulas trivialising (NC)

Given a particular logic, we define three classes of formulas each of whose members will trivialise (NC) if they are tautologies of that logic.

Let ψ be any arbitrary but fixed formula containing no atomic propositions other than p and q. (Sometimes we will write "$\psi(p,q)$" making the atomic formulas p and q explicit. The formula $\psi(A,B)$ is found by substituting A for p and B for q in ψ.) For each ψ, let us define two ancillary classes of formulas, V_ψ and W_ψ, as follows:

$$\begin{aligned} V_\psi &= \{p \to \psi(p,q), \psi(p,q) \to p\} \\ W_\psi &= \{\psi(p,q) \to (p \to q), p \to (\psi(p,q) \to q)\} \end{aligned}$$

Now let us define the three classes X_ψ, Y_ψ and Z_ψ. Members of each class will trivialise (NC). Each class is not only relative to a choice of ψ, but also to the logic under consideration, in a way which will become clear given the definitions. For these definitions we need yet another notation. For the repeated conditional

$$A_n \to (\cdots \to (A_2 \to (A_1 \to B))\cdots)$$

we will write "$\Pi_{i=1}^n A_i \to B$." This notation will compress the presentation of repeated conditionals.[1] The number n is the *depth* of the formula $\Pi_{i=1}^n A_i \to B$. Now, on to the definitions of classes:

$$\begin{aligned} X_\psi &= \{\Pi_{i=1}^n A_i \to q : n \geq 0 \text{ and } A_i \in V_\psi \text{ or a tautology}\} \\ Y_\psi &= \{\Pi_{i=1}^n A_i \to \psi(p,q) : n \geq 0 \text{ and } A_i \in V_\psi \text{ or a tautology}\} \\ Z_\psi &= \{\Pi_{i=1}^n A_i \to p : n \geq 0 \text{ and } A_i \in V_\psi \text{ or a tautology}\} \end{aligned}$$

We note that each of our examples in sections 2, 3 and 4 falls into one of these general classes of formulas. A, L, SC and their extensions are members of the class X_ψ, W and its extensions are members of the class Y_ψ, and PL and its extensions are members of the class Z_ψ. For example, choosing $\psi(p,q)$ to be $q \to p$ gives us A as $((\psi(p,q) \to p) \to q)$.

Lemma 1 *For (i) any formula $R \in X_\psi$, or (ii) any pair of formulas $R \in Y_\psi$ and $S \in W_\psi$, or (iii) any formula $R \in Z_\psi$, where the depth of the formula is 0 or each A_i is a tautology then that logic will be trivial.*

This proof is in two cases, one for $n = 0$ and the other for each A_i a tautology. Note that for $n = 0$ any formula in any of these classes is enough to trivialise a logic without appealing to (NC$\to$) or (NC$\leftarrow$).

Proof. *Case 1*: $n = 0$

Case 1 (i): $R \in X_\psi$. In this case R is just the formula q.

[1] Note, $\Pi_{i=1}^n A_i$ is not a formula.

Case 1 (ii): $R \in Y_\psi$. This case splits into two subcases.

Case 1 (ii) a: $R = \psi(p,q)$ and $S = \psi(p,q) \to (p \to q)$. The triviality proof is this:

(1)	$\psi(p,q)$	R
(2)	$\psi(p,q) \to (p \to q)$	S
(3)	$p \to q$	1, 2
(4)	$\psi(p,q) \to q$	substitute $\psi(p,q)$ for p in (3)
(5)	q	1, 4 MP

Case 1 (ii) b: $R = \psi(p,q)$, $S = p \to (\psi(p,q) \to q)$. The proof is similar:

(1)	$\psi(p,q)$	R
(2)	$p \to (\psi(p,q) \to q)$	S
(3)	$\psi(p,q) \to (\psi(\psi(p,q),q) \to q)$	substitute $\psi(p,q)$ for p in (2)
(4)	$\psi(\psi(p,q),q) \to q$	1, 3 MP
(5)	$\psi(\psi(p,q),q)$	substitute $\psi(p,q)$ for p in (1)
(6)	q	4, 5 MP

Case 1 (iii): $R \in Z_\psi$. Then R is just p.

Case 2: each A_i a tautology.

Case 2 (i): X_ψ: $\Pi_{i=1}^{n} A_i \to q$. If each A_i is a tautology, then n applications of MP will yield q.

Case 2 (ii): Y_ψ: $\Pi_{i=1}^{n} A_i \to \psi(p,q)$. If each A_i is a tautology, then n applications of MP will yield $\psi(p,q)$. From here the proof is the same as for *Case 1 (ii)* where $n = 0$.

Case 2 (iii): Z_ψ: $\Pi_{i=1}^{n} A_i \to p$. Now n applications of MP yields p and triviality is immediate. ■

Lemma 2 *Any substitution instance where we substitute $a \in a$ for p in (i) any formula $R \in X_\psi$, (ii) any pair of formulas $R \in Y_\psi$ and $S \in W_\psi$ or (iii) any pair of formulas $R \in Z_\psi$ and $S \in W_\psi$ will trivialise (NC).*

Proof. We proceed by induction on n, the depth of the formula R. Recall the formulas (NC):

(NC→) $\quad a \in \{x : \varphi(x)\} \to \varphi(a)$
(NC←) $\quad \varphi(a) \to a \in \{x : \varphi(x)\}$

Now take $\psi = \psi(p,q)$ as above and, writing $\psi(a \in a, q)$ for the result of substituting '$a \in a$' for all occurrences of p, consider the special case of (NC→) and (NC←) in which $\varphi(t)$ is $\psi(t \in t, q)$. Then let $a = \{x : \varphi(x)\}$ and substitute a for t in (NC→) and (NC←). We have the instances:

(NC→) $\quad a \in a \to \psi(a \in a, q)$
(NC←) $\quad \psi(a \in a, q) \to a \in a$

Let Q' be the substitution instance of Q that is the result of substituting $a \in a$ for p throughout. The proof now proceeds by the three cases stated.

Case (i): $R \in X_\psi$. Let n be the depth of R. The base case of our induction is $n = 0$, and this is covered by Lemma 1.

Now suppose $n = m + 1$, and that the lemma holds for formulas of depth m. Then $R = \Pi_{i=1}^{m+1} A_i \to q$. Now consider what the formula A_{m+1} might be.

Case (i) a: A_{m+1} is $p \to \psi(p, q)$. Now substitute $a \in a$ for p throughout R. We can prove triviality as follows:

(1)	$(a \in a \to \psi(a \in a, q)) \to \Pi_{i=1}^{m} A_i' \to q$	R'
(2)	$a \in a \to \psi(a \in a, q)$	NC $\to$
(3)	$\Pi_{i=1}^{m} A_i' \to q$	1, 2 MP

Now (3) is a substitution instance (substituting $a \in a$ for p throughout) of the formula $Q = \Pi_{i=1}^{m} A_i \to q$, where $Q \in X_\psi$ and has depth m, so by the induction hypothesis, we have triviality.

Case (i) b: A_{m+1} is $\psi(p, q) \to p$. As before, substitute $a \in a$ for p throughout R.

(1)	$(\psi(a \in a, q) \to a \in a) \to \Pi_{i=1}^{m} A_i' \to q$	R'
(2)	$\psi(a \in a, q) \to a \in a$	NC $\leftarrow$
(3)	$\Pi_{i=1}^{m} A_i' \to q$	1, 2 MP

and as in *Case (i) a*, (3) is a substitution instance (substituting $a \in a$ for p throughout) of the formula $Q = \Pi_{i=1}^{m} A_i \to q$, where $Q \in X_\psi$ and has depth m so by hypothesis of induction, we have triviality.

Case (i) c: A_{m+1} is a tautology.

(1)	$A_{m+1} \to \Pi_{i=1}^{m} A' \to q$	R'
(2)	A_{m+1}	tautology
(3)	$\Pi_{i=1}^{m} A' \to q$	1, 2 MP

The rest of the proof is as before, appealing to the induction hypothesis.

Case (ii): $R \in Y_\psi$ and $S \in W_\psi$. As before, let n be the depth of R. The base case of our induction is $n = 0$, and this is covered by Lemma 1.

Now suppose $n = m + 1$, and that the lemma holds for formulas of depth m. Then $R = \Pi_{i=1}^{m+1} A_i \to \psi(p, q)$, and either $S = \psi(p, q) \to (p \to q)$ or $S = p \to (\psi(p, q) \to q)$.

Case (ii) a: A_{m+1} is of the form $p \to \psi(p, q)$. Then substitute $a \in a$ for p throughout R to form R'. We have this proof:

(1)	$(a \in a \to \psi(a \in a, q)) \to (\Pi_{i=1}^{m} A' \to \psi(a \in a, q))$	R'
(2)	$a \in a \to \psi(a \in a, q)$	NC $\to$
(3)	$\Pi_{i=1}^{m} A' \to \psi(a \in a, q)$	1, 2 MP

(3) is a substitution instance (obtained by substituting $a \in a$ for p throughout) of the formula $Q = \Pi_{i=1}^{m} A_i \to \psi(p, q)$. $Q \in Y_\psi$ and has depth m, so by hypothesis of induction, we have triviality.

Case (ii) b: A_{m+1} is of the form $\psi(p,q) \to p$. The proof proceeds in the same way as for *Case (ii) a*: In this instance at line (2) we would use (NC$\leftarrow$) instead of (NC$\to$).

Case (ii) c: A_{m+1} is a tautology.

(1)	$A'_{m+1} \to (\Pi_{i=1}^{m} A' \to \psi(a \in a, q))$	R'
(2)	A'_{m+1}	tautology
(3)	$\Pi_{i=1}^{m} A' \to \psi(a \in a, q)$	1, 2 MP

(3) is a substitution instance of the formula $Q = \Pi_{i=1}^{m} A_i \to \psi(p,q)$. $Q \in Y_\psi$ and has depth m, so by hypothesis of induction, we have triviality.

Case (iii): $R \in Z_\psi$ and $S \in W_\psi$. As before, let n be the depth of R. The base case of our induction is $n = 0$. Then R is just p and either $S = \psi(p,q) \to (p \to q)$ or $S = p \to (\psi(p,q) \to q)$. Let's take these two cases in turn.

Case (iii) a: $S = \psi(p,q) \to (p \to q)$

(1)	$a \in a$	R'
(2)	$\psi(a \in a, q) \to (a \in a \to q)$	S'
(3)	$a \in a \to \psi(a \in a, q)$	NC$\leftarrow$
(4)	$\psi(a \in a, q)$	1, 3 MP
(5)	$a \in a \to q$	2, 4 MP
(6)	q	1, 5 MP

Case (iii) a: $S = p \to (\psi(p,q) \to q)$

(1)	$a \in a$	R'
(2)	$a \in a \to (\psi(a \in a, q) \to q)$	S'
(3)	$\psi(a \in a, q) \to q$	1, 2 MP
(4)	$a \in a \to \psi(a \in a, q)$	NC$\leftarrow$
(5)	$\psi(a \in a, q)$	1, 4 MP
(6)	q	3, 5 MP

Now suppose that $n = m + 1$, and that the lemma holds for formulas of depth m. Then $R = \Pi_{i=1}^{m+1} A_i \to p$, and either $S = \psi(p,q) \to (p \to q)$ or $S = p \to (\psi(p,q) \to q)$.

Case (iii) a: A_{m+1} is of the form $p \to \psi(p,q)$. In this case the proof proceeds in the same way as for *Case (i) a* except that the formula at line (3) is a substitution instance of $Q = \Pi_{i=1}^{m} A_i \to p$ and $Q \in Z_\psi$ and not X_ψ.

Case (iii) b: A_{m+1} is of the form $\psi(p,q) \to p$. In this case the proof proceeds in the same way as for *Case (i) b* with the same change made as for *Case (iii) a* .

Case (iii) c: A_{m+1} is a tautology. The same applies in this case. ∎

Theorem 3 *For any given choice of ψ (i) any formula $R \in X_\psi$ (ii) any pair of formulas $R \in Y_\psi$ and $S \in W_\psi$ or (iii) any pair of formulas $R \in Z_\psi$ and $S \in W_\psi$ provable in a logic will trivialise (NC).*

Proof. The proof is an induction on n, the depth of the formula R. The base case, $n = 0$, is dealt with by Lemma 1. For the induction step, take $n = m + 1$.

Case (i): $R \in X_\psi$. Then $R = \Pi_{i=1}^{m+1} A_i \rightarrow q$. Choose a substitution instance of R by substituting $a \in a$ for $p.(\acute{R})$. Then by Lemma 5.2, $\acute{R}$ trivialises naïve comprehension.

Case (ii) and (iii). These cases proceed as for Case (i). ■

It is worth noting that each of these general classes of formulas has members that are classical tautologies. This is obvious in the cases of Y_ψ and Z_ψ, and for the class X_ψ we have $((p \rightarrow q) \rightarrow p) \rightarrow ((p \rightarrow (p \rightarrow q)) \rightarrow q)$.[2]

6 Furthermore…

So far we have mentioned the axioms W, A, SC, L and PL. In this section we shall show that logics including axioms from among B, C, K and B´ can be trivialised in even more ways.

B	$(p \rightarrow r) \rightarrow ((q \rightarrow p) \rightarrow (q \rightarrow r))$
C	$(p \rightarrow (q \rightarrow r)) \rightarrow (q \rightarrow (p \rightarrow r))$
K	$p \rightarrow (q \rightarrow p)$
B´	$(p \rightarrow q) \rightarrow ((q \rightarrow r) \rightarrow (p \rightarrow r))$

(Note that B´ is derivable from B and C.)

Lemma 4 *From any* $m \rightsquigarrow n$, $m > n$, *contracting implication in a logic with both K and either B´or B and C is also an* $n + 1 \rightsquigarrow n$ *contracting implication.*

Proof. Consider any $m \rightsquigarrow n$, $m > n$ contracting implication. We will show that if $m > n+1$, this implication is also an $m-1 \rightsquigarrow n$ contracting implication. Repeat this result as necessary until you get to $n+1 \rightsquigarrow n$. In this proof, we will write "A^m" for $p \;{}_{[m]}\!\!\rightarrow q$, to save space and to make the structure clear. The crucial step is to note that $A^{m-1} \rightarrow A^m$ — that is, $(p \;{}_{[m-1]}\!\!\rightarrow q) \rightarrow (p \;{}_{[m]}\!\!\rightarrow q)$, or even more perspicuously, $(p \;{}_{[m-1]}\!\!\rightarrow q) \rightarrow (pimp(p \;{}_{[m-1]}\!\!\rightarrow q))$ — is an instance of K.

(1)	$A^m \rightarrow A^n$	$m \rightsquigarrow n$
(2)	$A^{m-1} \rightarrow A^m$	K
(3)	$(A^{m-1} \rightarrow A^m) \rightarrow ((A^m \rightarrow A^n) \rightarrow (A^{m-1} \rightarrow A^n))$	B′ ■
(4)	$(A^m \rightarrow A^n) \rightarrow (A^{m-1} \rightarrow A^n)$	2, 3 MP
(5)	$A^{m-1} \rightarrow A^n$	1, 4 MP

This lemma, together with the Shaw-Kwei result [see section 4.1] prove the following corollary:

[2] We thank Lloyd Humberstone for this example.

Corollary 5 *Any logic, with K and B' with an $m \rightsquigarrow n$, $m > n$, contracting implication trivialises (NC).*

White [10] has shown that Łukasiewicz' infinite-valued predicate logic is consistent with (NC). Therefore this result together with corollary 6.2 gives us:

Corollary 6 *There is no contracting implication, $m \rightsquigarrow n$, $m > n$, definable in the language of Łukasiewicz' infinite-valued predicate logic.*

BCK logic is a sub-logic of Łukasiewicz' infinite-valued predicate logic and therefore is also consistent with (NC). Thus, these corollaries together give:

Corollary 7 *There is no contracting implication, $m \rightsquigarrow n$, $m > n$, definable in the language of BCK logic.*

7 Extension of Restall's Result

We have generalised each of our examples above by giving a class of formulas such that any member of any of these classes trivialises (NC). These results can be further generalised in a different way. Greg Restall [6] showed that we could generalise Curry's result to include any contracting implication definable in the language of the logic. That is, even if a logic was contraction free, if it was possible to define a connective, $>$, in the language of that logic that satisfied the following conditions:

[C1] $A \to B \vdash A > B$

[C2] $A, A > B \vdash B$

[C3] $A > (A > B) \vdash A > B$

then the logic trivialises (NC). We can extend Restall's result to incorporate our results from Section 5.

Let$\psi = \psi(p, q)$ be any arbitrary but fixed formula in at most p and q as before. Consider the ancillary class of formulas, V^*_ψ, defined as follows:

$$V^*_\psi = \{p > \psi(p, q), \psi(p, q) > p\}$$

Let us define a further ancillary class of formulas, W^*_ψ:

$$W^*_\psi = \{\psi(p, q) > (p > q), p > (\psi(p, q) > q)\}$$

Now, taking "$\Pi^n_{i=1} A_i > B$" to be the obvious shorthand for an n-fold nested $>$-implication, let us define three general classes of formulas as follows:

$$\begin{aligned}
X^*_\psi &= \{\Pi^n_{i=1} A_i > q : n \geq 0 \text{ and } A_i \in V^*_\psi \text{ or a tautology}\} \\
Y^*_\psi &= \{\Pi^n_{i=1} A_i > \psi(p, q) : n \geq 0 \text{ and } A_i \in V^*_\psi \text{ or a tautology}\} \\
Z^*_\psi &= \{\Pi^n_{i=1} A_i > p : n \geq 0 \text{ and } A_i \in V^*_\psi \text{ or a tautology}\}
\end{aligned}$$

Theorem 8 *For any given choice of ψ, (i) any formula $R \in X^*_\psi$ (ii) any pair of formulas $R \in Y^*_\psi$ and $S \in W^*_\psi$ or (iii) any pair of formulas $R \in Z^*_\psi$ and $S \in W^*_\psi$ provable in a logic will trivialise (NC).*

Proof. This is proved by induction on n. The proof proceeds in much the same way as that in section 5. The base case is given by the following lemma. ■

Lemma 9 *For any formula $R \in X^*_\psi$ or $R \in Y^*_\psi$ or $R \in Z^*_\psi$ provable in a given logic, that logic will be trivial when (i) $n = 0$ or (ii) when each A_i is a tautology.*

Proof. This is as in Lemma 1, except instead of applying MP we appeal to C2. We have as an instances of (NC$\rightarrow$) and (NC$\leftarrow$), $\psi(a \in a, q) \rightarrow a \in a$ and $a \in a \rightarrow \psi(a \in a, q)$ Whenever either direction of (NC) is used, in the next step of the proof we appeal to C1 to replace the $\rightarrow$ with $>$. In subsequent steps of the proof C2 is appealed to rather than MP. ■

8 Further Research Directions

These results exhaust the class of implicational formulas *known* to trivialise (NC). There may be others, but this remains to be shown. When we introduce other connectives to the language, the situation is different. There are many more kinds of formulas which trivialise (NC). A systematic discussion of these is beyond the scope of this paper, but we can indicate just a few of the phenomena which may be observed. Martin Bunder [1] gives examples of sets of tautologies that trivialise (NC) which, instead of relying on a formula, $S \in W_\psi$, rely on formulas of the form

1. $((p \rightarrow q) \wedge p) \rightarrow q$ (that is, the *modus ponens* axiom), and
2. $((p \circ p) \rightarrow r) \rightarrow (p \rightarrow r)$.

The contraction-like properties of these have been well known for some time [5]. These formulas trivialising (NC) involve connectives other than $\rightarrow$. All of the routes to triviality we know which use $\rightarrow$ alone are covered by our general classes.

The class of formulas, W_ψ, could perhaps be generalised in some way to include these examples. This is not to say that a formula $S \in W_\psi$ (as defined above) could not be derived from a set of tautologies that seemingly rely on (1) and (2) above in order to trivialise (NC).

We have shown that there are many different routes to triviality. Some, like contraction are well worn; others are much less known. For anyone interested in the inconsistencies arising out of the paradoxes of naïve comprehension, we hope to have at least provided a more comprehensive map of the terrain.

9 References

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